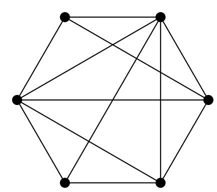
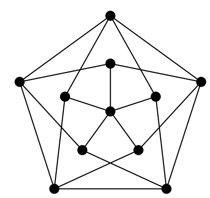
Graph Theory - Problem Set 5 (Solutions)

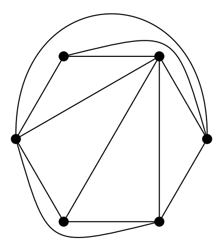
October 10, 2024

Exercises

1. Determine if the following graphs are planar or not.

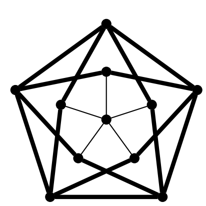



Solution. The graph on the left is planar, since it has a planar drawing, for example:



The graph on the right is not planar. Suppose not, then by Euler's formula any planar drawing of it would have $|F(G)| = 2 - |V(G)| + |E(G)| = 2 - 11 + 20 = 11$ faces. However one can observe that the graph contains no triangles, so all faces would be bordered by ≥ 4 edges. Each edge borders 2 faces (there are no cut-edges in this graph), so we should have $2e \geq 4f$ which implies that $40 \geq 44$, a contradiction.

Alternatively, one can observe that the graph contains a subdivision of K_5 as follows.



2. Determine all positive integers r and s for which $K_{r,s}$ is planar.

Solution. First, note that if $r, s \geq 3$, then $K_{r,s}$ is not planar, since it contains $K_{3,3}$ as a subgraph. On the other hand, $K_{1,s}$ and $K_{2,s}$ are planar for all $s \geq 1$, since the first one can be easily drawn by non-crossing straight line edges as a star graph, and a planar drawing of the second one is similar: put s vertices on a line in the plane, and add two extra vertices, one on each side of the line. Then draw edges as straight line segments.

3. Let G be a graph on n vertices and $3n-6+k$ edges for some $k>0$. Show that any drawing of G in the plane contains at least k crossing pairs of edges.

   Solution. We prove it by induction on k. For the induction basis $k=1$, one knows that a planar graph on n vertices has at most $3n-6$ edges, so any drawing of a graph with $3n-5$ edges has at least a pair of crossing edges. Now suppose the statement holds for k, we prove it for $k+1$: By the same reason as before, any drawing D_G of G has at least a pair of crossing edges, e, e'. Now remove e from the drawing, this gives a drawing of a graph with $3n-6+k$ edges which by the induction hypothesis has at least k pair of crossing edges. So altogether, the drawing D_G has at least $k+1$ crossings.

4. Let G be a planar graph with fewer than 12 vertices. Show that G has a vertex of degree at most 4.

   Solution. By the corollary of Euler's formula, we have

$$\sum_{v\in V(G)} \deg(v) = 2|E(G)| \leq 2(3n-6) = 6n-12 < 6n-n = 5n,$$

   which by the pigeonhole principle implies that G has a vertex of degree at most 4.

Problems

5. Using Euler's formula, show that if G is a planar graph on n vertices that has finite girth g, then G has at most $\frac{g}{g-2}(n-2)$ edges. Deduce that the Petersen graph (see below) is not planar.

   Solution: The proof goes in the same direction as the proof of the following result shown in the lecture: every planar graph on n vertices has at most $3n-6$ edges. First, we can assume that G is connected, since otherwise one can add extra edges to make it connected. Fix the planar drawing D of G. By double counting the pairs (e, f) where e is an edge on the boundary of face f, we get $2|E(G)| \geq g|F_D(G)|$. Plugging in the Euler's formula, we get

$$|V(G)| - |E(G)| + \frac{2}{g}\cdot|E(G)| \geq 2 \Rightarrow |E(G)| \leq \frac{g}{g-2}(n-2).$$

   For the Petersen graph, use the same argument by noting that the girth is 5, the number of vertices is 10 and it has 15 edges.

6. (a) Let G be a planar graph containing no triangles. Show that $\chi(G) \leq 4$.

   (b) Let G be a planar graph containing at most three triangles. Show that $\chi(G) \leq 4$.

   Solution:

   (a) We prove by induction on the number of vertices of G. If $|V(G)| \leq 4$, there is nothing to prove. Suppose that the statement holds for all the graphs with $n-1$ vertices, we prove it for the graph G on n vertices.

   First, we show that G has a vertex of degree at most 3. Since the girth of G is at least 4, by following the same proof of the previous exercise, we get $|E(G)| \leq \frac{4}{2}(n-2) = 2n-4$. So we have

$$\sum_{v\in V(G)} \deg(v) = 2|E(G)| \leq 4n-8,$$

which by the pigeonhole principle implies that there exists $v \in V(G)$ with $\deg(v) \leq 3$. Now by the induction hypothesis, since the graph $G - v$ is also triangle-free, we have $\chi(G - v) \leq 4$. So by coloring the vertex v by a color different from its neighbors, we get a valid 4-coloring of G.

(b) The proof is similar to the proof of part (a). Again we prove by induction on the number of vertices. For a graph G on n vertices with at most three triangles, if we remove one edge from each triangle, then the resulting graph becomes triangle-free, so by the previous exercise, we have $|E(G)| \leq 2n - 4 + 3 = 2n - 1$, which implies that G has a vertex of degree at most 3. The rest of the proof is the same as before.

7. Prove that for any three vertices x, y, z of a planar graph on n vertices, the sum of the degrees $d(x) + d(y) + d(z)$ is at most $2n + 2$.

   Hint: Use the fact that a planar graph does not contain $K_{3,3}$ as subgraph.

   Solution: Denote the graph by G. First note that if x, y, z are pairwise connected, each of the three pairwise edges among them appears twice in the sum $d(x) + d(y) + d(z)$. On the other hand, note that since G cannot have $K_{3,3}$ as a subgraph, at most two vertices of $G \setminus \{x, y, z\}$ are connected to all x, y, z. This means that the all $(n-3)$ vertices of $G \setminus \{x, y, z\}$, probably except two, are connected to at most two vertices among $\{x, y, z\}$. Therefore, we get:

$$d(x) + d(y) + d(z) \leq 2(n - 5) + 3 \cdot 2 + 6,$$

   where 6 accounts for the maximum number of pairwise edges between x, y, z, as discussed in the beginning.

8. Let S be a set of n points in the plane such that any two of them have distance at least 1. Show that there are at most $3n - 6$ pairs of distance *exactly* 1.

   Hint: Prove that the graph has no crossing using the triangle inequality.

   Solution: For any pair of points $p, q \in S$, we connect them by their connecting line segment if $d(p, q) = 1$, where $d(p, q)$ denotes their distance. The resulting drawing gives us a graph. We only need to show that this graph has no crossing. Then the result follows from the corollary of Euler's formula.

   Now suppose by contradiction that for $p, q, r, s \in S$, the edges pq, rs cross at x. Let p, r, q, s appear in this order on the boundary of their convex hull. By the triangle inequality we have

$$d(p, s) < d(p, x) + d(x, s), \;\; d(q, r) < d(q, x) + d(r, x),$$

   which implies that $2 \leq d(p, s) + d(q, r) < d(p, q) + d(r, s) = 2$, which is a contradiction.