Graph Theory - Problem Set 4 (Solutions)

October 3, 2024

Exercises

1. In this exercise, we show that the sufficient conditions for Hamiltonicity that we saw in the lecture are tight in some sense.

   (a) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has $\binom{n-1}{2} + 1$ edges.

   (b) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has minimum degree $\lceil \frac{n}{2} \rceil - 1$.

   (c) Find a non-Hamiltonian graph G which satisfies $\alpha(G) = \kappa(G) + 1$.

   Solution.

   (a) Consider the complete graph on $n-1$ vertices K_{n-1}. Add a new vertex v and connect it to a vertex of K_{n-1}. This graph has $\binom{n-1}{2} + 1$ edges and it is not Hamiltonian: every cycle uses 2 edges at each vertex, but v has only one adjacent edge.

   (b) Let G_1 be a complete graph on $\lceil \frac{n}{2} \rceil$ vertices and G_2 be a complete graph on $\lfloor \frac{n}{2} \rfloor$ vertices which is disjoint from G_1. Fix a vertex $v \in V(G_1)$ and connect it to all the vertices of G_2. Let G be the resulting graph. Then G has minimum degree $\lceil \frac{n}{2} \rceil - 1$ and it is non-Hamiltonian: every cycle passing through all the vertices of G has to pass through v at least twice.

   (c) Consider the bipartite graph $K_{n,n+1}$. Note that $\alpha(K_{n,n+1}) = n+1$ and $\kappa(K_{n,n+1}) = n$, but this graph is non-Hamiltonian since after deleting all n vertices of one side there are $n+1$ connected components.

   Alternatively, one can take the Petersen graph as an example, whose indepedence number is 4 and connectivity is 3.

2. For every $k, n \geq 2$, find a graph G on *at least* n vertices such that $\delta(G) = k$ but G contains no cycle longer than $k+1$.

   Solution. Let $a = \lceil n/k \rceil$, and take a disjoint copies of K_{k+1}. Then this graph has $a(k+1) > n$ vertices, where each of them has degree k, but there is no cycle longer than $k+1$. We can actually find a connected such graph by joining all of these cliques at one vertex (add more copies of cliques if the degree of the centered vertex is less than k), to get a star of cliques.

3. Check that the proof of Dirac's theorem also proves Ore's theorem.

   Dirac's theorem: Let G be a graph on $n \geq 3$ vertices. If $\delta(G) \geq \frac{n}{2}$, then G contains a Hamilton cycle.

   Ore's theorem: Let G be a graph on $n \geq 3$ vertices. If $d(u) + d(v) \geq n$ for any non-adjacent vertices u and v, then G contains a Hamilton cycle.

   Solution. There are two places in the proof of Dirac's Theorem where we use the condition that $\delta(G) \geq \frac{n}{2}$:

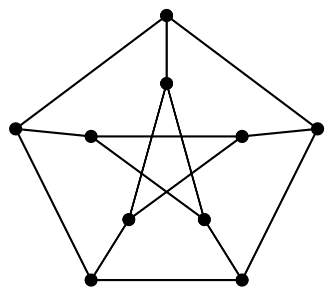
(1) to show that G is connected,

(2) to show that there is an edge in P that is both type-1 and type-2.

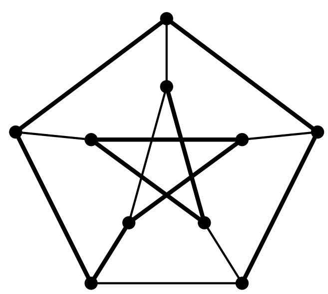
The proof of the first statement is very similar in this case: if u and v were in different components, then the component of u would contain at least $d(u) + 1$ vertices, and the component of v would have at least $d(v) + 1$ vertices, which would give more than n vertices in total.

The second statement follows because there are $d(v_1)$ type-1 edges and $d(v_k)$ type-2 edges. But then if v_1 and v_k are not adjacent, then P has $n-1 < d(v_1) + d(v_k)$ edges by assumption, so some edge is both type-1 and type-2 and we can continue the argument. Otherwise, we get a cycle $v_1 \dots v_k v_1$ that we can use as C for the rest of the proof.

4. The graph below is called the Petersen graph. Does it have a Hamilton path? Does it have a Hamilton cycle? Provide reasons for your answers.



Solution. A Hamilton path is given as follows.



There is no Hamilton cycle in the Petersen graph. To prove it, first one can check that the girth of P is 5. Therefore, there is no 3-cycle or 4-cycle in the Petersen graph. Also one can observe that the degree of each vertex is 3.

Suppose that there is a Hamilton cycle C in the Petersen graph, which is a 10-cycle. To get the Petersen graph, we need to add 5 edges to the cycle C. Since each vertex has degree 3 in the Petersen graph, we need to add exactly one edge for each vertex.

If each of the latter edges connects two opposite vertices on C, then there is a 4-cycle, a contradiction. Otherwise, some edge e joins vertices at distance 4 in C (why cannot it be 2 or 3?). Let e be incident to vertices A and B, and D be the opposite vertex to A in C. The vertex D must be connected to one of the neighbours of A in C (Why?), let us call it F. Then $ABDFA$ is a 4-cycle, a contradiction.

Problems

5. Use Ore's theorem from Exercise 3 to give a short proof of the fact that any n-vertex graph G with more than $\binom{n-1}{2}+1$ has a Hamilton cycle.

   Solution. If the graph is complete then it has a Hamilton cycle. Othweise, for any two non-adjacent vertices u, v, remove them from G to get a graph $G-u-v$ with $n-2$ vertices and $|E(G)|-d(u)-d(v)$ edges. Since $G-u-v$ has at most $\binom{n-2}{2}$ edges, we have

   $$\binom{n-2}{2} \geq |E(G)| - d(u) - d(v) > \binom{n-1}{2} + 1 - d(u) - d(v)$$

   $$\implies \quad d(u)+d(v) > \binom{n-1}{2} - \binom{n-2}{2} + 1 = n-1.$$

   Thus the condition of Ore's theorem holds, so G has a Hamilton cycle.

6. Let G be a connected graph on n vertices with minimum degree δ. Show that

   (a) if $\delta \leq \frac{n-1}{2}$ then G contains a path of length 2δ, and

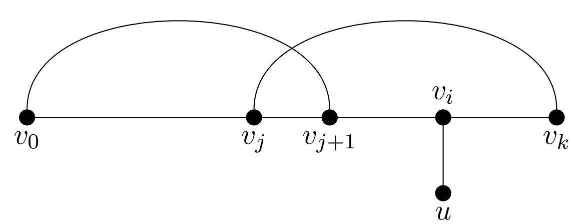
   (b) if $\delta \geq \frac{n-1}{2}$ then G contains a Hamilton path.

   Solution. It suffices to show that G contains a path of length at least $\min\{2\delta, n-1\}$.

   Take a longest path $P = v_0 v_1 \cdots v_k$ in G. If $k = n-1$ we are done, so we can assume that there is a vertex not on P. By connectedness of G, there must be a vertex u that is not in P but adjacent to a vertex of P, say v_i. We have the following observations:

   - v_i cannot be v_0 or v_k, otherwise we could extend P to get a longer path.
   - v_0 and v_k cannot be adjacent, otherwise $uv_i \cdots v_k v_0 \cdots v_{i-1}$ would be a longer path.
   - $N(v_0)$ and $N(v_k)$ must be contained in $\{v_1, \dots, v_k\}$, otherwise we could the extend P.

   One more observation which is a bit trickier: we cannot have anything like (not showing all the vertices on P)

   since this would also give a longer path:

   $$v_{i+1} \cdots v_k v_j \cdots v_0 v_{j+1} \cdots v_i u.$$

   Note that something similar works when $j > i$. Thus we cannot have v_j adjacent to v_k and at the same time v_{j+1} adjacent to v_0.

   The set $\{v_1, \dots, v_k\}$ contains the two sets $N(v_0)$ and $\{v_{j+1} : v_j \in N(v_k)\}$, both of which have size at least δ. Our last observation implies that these two sets are disjoint, which tells us that $k \geq 2\delta$.

7. Suppose that each edge of the complete graph K_n is painted either red or blue. Prove that this colored graph has a Hamilton path, which is the union of a red path and a blue path. (We allow the case when one of the paths has length 0, i.e., the Hamilton path uses only one color.)

   Solution. Take a longest such path $P = v_k \dots v_1 w u_1 \dots u_l$, where $v_k \dots v_1 w$ is a red path and $w u_1 \dots u_l$ is a blue path. If P is not Hamiltonian, then there is a vertex x not contained in it. Look at the edge wx. If it is red, then the path $v_k \dots v_1 w x u_1 \dots u_l$ satisfies the required property, and it is longer (no matter if xu_1 is red or blue). If wx is blue, then $v_k \dots v_1 x w u_1 \dots u_l$ is a longer such path, a contradiction.

   Alternatively, one can prove this statement by induction on n, using the same idea of looking at wx.

8. Prove the following sufficient conditions of Hamiltonicity, which generalizes Dirac's theorem.

   Chvátal's theorem (1972): Let G be a graph on $n \geq 3$ vertices, whose degree sequence of its vertices is $d_1 \leq d_2 \leq \dots \leq d_n$. If there is no $1 \leq k < \frac{n}{2}$ for which $d_k \leq k$ and $d_{n-k} < n-k$, then G is Hamiltonian.

   Hint: Show that its closure $c(G)$ is complete by contradiction. Start by taking two non-adjacent vertices in $c(G)$ with the largest sum of degrees.

   Solution. Denote the degree of v in G as $d(v)$ and the degree of v in $c(G)$ as $d'(v)$.

   By what we have shown in the lecture, it suffices to show that the closure $c(G)$ is complete. Suppose not. Then take a pair of non-adjacent vertices u and v in $c(G)$ with

   $$d'(u) \leq d'(v) \tag{1}$$

   and their sum of degrees being the largest among all non-adjacent pairs. By definition of $c(G)$, we have

   $$d'(u) + d'(v) < n. \tag{2}$$

   Let S be the set of vertices in $V \setminus \{v\}$ which are non-adjacent to v in $c(G)$ and T be the set of vertices in $V \setminus \{u\}$ which are non-adjacent to u in $c(G)$. We have

   $$|S| = n - 1 - d'(v), \quad |T| = n - 1 - d'(u).$$

   By maximality of $d'(u) + d'(v)$, each vertex in S must have degree at most $d'(u)$ and each vertex in $T \cup \{u\}$ must have degree at most $d'(v)$. Define $k := d'(u)$. From (1) and (2) we get $k < \frac{n}{2}$. Moreover in $c(G)$ there are at least $|S| = n - 1 - d'(v) > k - 1$ (i.e., at least k) vertices of degree at most k and there are at least $|T \cup \{u\}| = n - d'(u) = n - k$ vertices of degree at most $d'(v) < n - k$. Both statements hold in G as well, which implies that $d_k \leq k$ and $d_{n-k} < n - k$, a contradiction to the assumption. Thus $c(G)$ must be complete.

9. (*) Let G be a graph in which every vertex has odd degree. Show that every edge of G is contained in an even number of Hamilton cycles.

   Hint: Let $xy \in E(G)$ be given. The Hamilton cycles through xy correspond to the Hamilton paths in $G - xy$ from x to y. Consider the set $\mathcal{H}$ of all Hamilton paths in $G - xy$ starting at x, and show that an even number of these end in y. To show this, define a graph on $\mathcal{H}$ so that the desired assertion follows from the fact (proved in Problem Set 1): the number of odd-degree vertices in a graph is always even.

Solution. As suggested in the *Hint*, it suffices to show that for any edge xy in G, there are even number of Hamilton paths in $G - xy$ which starts at x and ends at y. Let $\mathcal{H}$ be the set of all Hamilton paths in $G - xy$ starting at x. Define a graph whose vertices are exactly elements of $\mathcal{H}$ and edges are defined as follows: any two Hamilton paths P and Q in $G - xy$ starting at x (call such path a Hamilton x-path), are connected as an edge if

$$P = xx_1 \cdots x_k, \quad Q = xx_1 \cdots x_i x_k x_{k-1} \cdots x_{i+1}, \quad \text{where } x_i \text{ is adjacent to } x_k.$$

Thus for any $P = xx_1 \cdots x_k$, it is connected to $d'(x_k) - 1$ many other Hamilton x-paths, where $d'(x_k)$ is the degree of x_k in $G - xy$. Note that in $G - xy$ only x and y have even degree, thus only Hamilton x-paths that ends at y have odd degree, which equals to $d'(y) - 1 = d(y) - 2$. Since the number of odd-degree vertices is even in any graph, the number of Hamilton paths in $G - xy$ which starts at x and ends at y, is also even.