Graph Theory - Problem Set 3 (Solutions)

September 26, 2024

Exercises

1. Show that a graph is connected if and only if it contains a spanning tree.

   Solution. If there is a spanning tree then the graph is clearly connected: for any vertices u and v, there will be a u-v path in the tree, hence in the graph, as well. If the graph is connected then the BFS algorithm finds a spanning tree, and this proves that a spanning tree exists.

2. For what values of n does the graph K_n contain an Euler trail? An Euler tour? A Hamilton path? A Hamilton cycle?

   Solution. Each vertices in K_n has degree equal to $n-1$.

   Euler trail: K_n for all odd $n \geq 1$ and K_2
   Euler tour: K_n for all odd $n \geq 1$.
   Hamilton path: K_n for all $n \geq 1$.
   Hamilton cycle: K_n for all $n \geq 1$ except for K_2.

3. (a) For what values of m and n does the complete bipartite graph $K_{m,n}$ contain an Euler tour?

   (b) Determine the length of the longest path and the longest cycle in $K_{m,n}$, for all m, n.

   Solution.

   (a) Since for connected graphs the necessary and sufficient condition is that the degree of each vertex is even, m and n must be even positive integers.

   (b) The length of the longest cycle is $2 \cdot \min\{m, n\}$: Any cycle must be even, and it must alternate vertices from the two sides. Thus it cannot be longer that twice the smaller side (and such a cycle clearly exists).

   By a similar reasoning, if $m = n$, the longest path contains all the $2m$ vertices, so its length is $2m - 1$; if $m \neq n$, the length of the longest path is $2 \cdot \min\{m, n\}$, starting and ending in the larger class.

4. (a) Find a graph such that every vertex has even degree but there is no Euler tour.

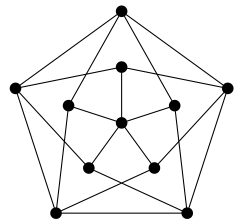
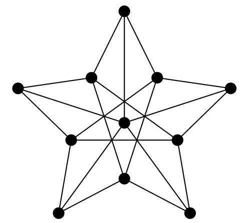
   (b) Find a disconnected graph that has an Euler tour.

   Solution.

   (a) Take a graph that is the vertex-disjoint union of two cycles. It is not connected, so there is no Euler tour.

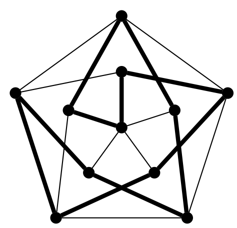
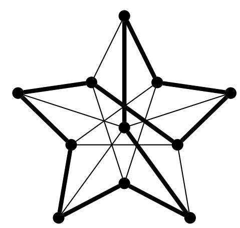
   (b) The empty graph on at least 2 vertices is an example. Or one can take any connected graph with an Euler tour and add some isolated vertices.

5. Determine the girth and circumference of the following graphs.

Solution. The graph on the left has girth 4: it's easy to find a 4-cycle and see that there is no 3-cycle. It has circumference 11: an 11-cycle of it is shown below, which is a Hamilton cycle.

The graph on the right also has girth 4 and circumference 11, by a similar argument (an 11-cycle of this graph is given below). Actually, one can check that these two graphs are isomorphic.

Problems

6. Let G be a connected graph that has an Euler tour. Prove or disprove the following statements.
   (a) If G is bipartite then it has an even number of edges.
   (b) If G has an even number of vertices then it has an even number of edges.
   (c) For edges e and f sharing a vertex, G has an Euler tour in which e and f appear consecutively.

   Solution.

   (a) True. The number of edges of a bipartite graph is equal to the sum of degrees of vertices in one part. If there is an Euler tour then all degrees are even, and the sum of even numbers is also even.
   (b) False. Take a cycle of length 3 and a cycle of length 4, joined at a single vertex. It has 6 vertices and 7 edges, and it has an Euler tour.
   Note: The sum of an even number of even numbers is not necessarily divisible by four!
   (c) False. The previous example of a graph works, just select e and f touching the common vertex, but from the same cycle.

   Remark (suggested by student). For (a), another proof is to use the following theorem (you need to prove this theorem before applying it):

   A connected graph G is Eulerian if and only if its edge set can be decomposed into cycles.

7. Show that if $k > 0$ then the edge set of any connected graph with $2k$ odd-degree vertices can be split into k trails. You have seen this problem in Problem Set 2 before. Here you are expected to find another solution.

   Hint: Add some edges to get a multigraph which has Euler tour. Then split the tour.

   Solution. Pair up the odd-degree vertices of the graph and add an edge between any pair. To distinguish them, let us say that the new edges are blue. The resulting *multi*graph is connected with all degrees even, so it has an Euler tour. Now delete the blue edges from this tour: we get k trails that cover each edge of the original graph exactly once.

8. Prove that in any connected graph G, there is a walk that uses each edge exactly twice.

   Solution. We duplicate each edge of G in order to get the new (connected) multigraph G'. Since all vertices of G' have even degree by construction, G' has an Euler trail. This gives the desired walk in G.

9. Let G be a connected graph with an even number of edges such that all the degrees are even. Prove that we can color each of the edges of G red or blue in such a way that every vertex has the same number of red and blue edges touching it.

   Solution. Let T be an Euler tour for G, which exists by the assumption. We get the desired coloring by coloring the edges of T (and as a result, the edges of G) alternatively by red and blue. Note that this coloring is feasible since G has even number of edges, and on the other hand, each vertex of G is adjacent to the same number of red and blue edges.