

Graph Theory - Problem Set 2 (Solutions)

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Exercises

1. Prove the triangle-inequality in graphs: for any three vertices u, v, w in a graph G ,

$$d(u, v) + d(v, w) \geq d(u, w).$$

Solution. If $d(u, v) = \infty$ or $d(v, w) = \infty$, there is nothing to prove.

Otherwise, according to the definition of the distance, there is a u - v path of the length $d(u, v)$ and a v - w path of the length $d(v, w)$. Join them together to obtain a u - w walk of the length $d(u, v) + d(v, w)$. We have proved in lecture that this u - w walk will contain a u - w path, which is clearly not longer than the walk. Therefore, the shortest u - w path has length at most $d(u, v) + d(v, w)$.

2. A graph that does not contain any cycles is called a *forest*.

Prove that a forest on n vertices with c connected components has exactly $n - c$ edges.

Solution. Let $T_1, \dots, T_c$ be the components of the forest, on $n_1, \dots, n_c$ vertices, respectively. Each T_i itself is a connected acyclic graph, hence it is a tree (considered as a graph on its own). Therefore, T_i contains $n_i - 1$ edges for each i . Altogether, the graph contains $\sum_{i=1}^c (n_i - 1) = \sum_{i=1}^c n_i - c = n - c$ edges.

3. Let T be a tree and e be an edge of T . Prove that $T - e$ is not connected.

Solution. Let $e = uv$ and suppose $T - e$ is connected. Then, in particular, $T - e$ contains a u - v path P . But then $P + e$ is a cycle in T , a contradiction.

4. Let T be a tree and let u and v be two non-adjacent vertices of T . Prove that $T + uv$ contains a unique cycle.

Solution. Since T is a tree, it is a connected graph without cycles. Then T contains a path between u and v and adding the edge uv gives a cycle in $T + uv$. For uniqueness, suppose by contradiction that $T + uv$ contains at least two cycles C_1 and C_2 . Since T does not contain a cycle, uv must be contained in both cycles. By removing uv from both cycles we get two different u - v paths in T , which contradicts the uniqueness of path in T .

Problems

5. Let G be a graph with minimum degree $\delta > 1$. Prove that G contains a cycle of length at least $\delta + 1$.

Solution. First, let's recall how we proceeded in the lecture to find a path of length at least δ : let $v_1 \cdots v_k$ be a maximal path in G , i.e., a path that cannot be extended. Then any neighbor of v_1 must be on the path, since otherwise we could extend it. Since v_1 has at least

$\delta(G)$ neighbors, the set $\{v_2, \dots, v_k\}$ must contain at least $\delta(G)$ elements. Hence $k \geq \delta(G) + 1$, so the path has length at least $\delta(G)$.

Now in order to find a cycle of length at least $\delta + 1$, we continue the proof above. The neighbor of v_1 that is furthest along the path must be v_i with $i \geq \delta(G) + 1$. Then $v_1 \cdots v_i v_1$ is a cycle of length at least $\delta(G) + 1$.

6. Show that every tree T has at least $\Delta(T)$ leaves.

Solution. Let v be a vertex with degree $d = \Delta(T)$. For every edge vw incident to v , take a longest path starting with vw . By maximality (as in the proof that every tree has a leaf), the last vertex of this path is a leaf. Doing this for each of the d edges incident to v , we get d paths starting at v , which are disjoint except for v (otherwise we would get a cycle). Thus each path gives a different leaf, and we get $d = \Delta(T)$ leaves.

Solution (alternative). If you remove v and its incident edges, you are left with d connected components $T_1, \dots, T_d$, each of which is a tree. By a lemma from class, every tree with at least two vertices has at least two leaves. Hence the T_i with at least two vertices have at least two leaves, one of which must be a leaf of T (one of the two leaves might have been adjacent to v , but not both because that would give a cycle). Some of the T_i might be single vertices, in which case those vertices were leaves in T (they must have been adjacent to v and to no other vertex).

7. Let G be a graph on n vertices. Prove that if G has at least $2n - 1$ edges, then it contains an even cycle.

Solution. In the lecture we proved that G contains a bipartite subgraph H with $|E(H)| \geq \frac{|E(G)|}{2}$. We have that $|V(H)| \leq n$ and $|E(H)| \geq n$ in H . Therefore, H contains a cycle (otherwise H is a forest, and the number of edges in a forest is strictly less than the number of vertices). Since H is a bipartite graph, this cycle has even length.

8. Let T be an n -vertex tree that has exactly $2k$ vertices of odd degree. Show that T can be split into k edge-disjoint paths, i.e., T is the union of k edge-disjoint paths. Note that we only consider path of length at least 1.

Hint: Prove by induction.

Solution. We prove a general statement for forest by induction on k . For $k = 0$ our forest has at most 1 vertex (every nonempty forest has a leaf, thus an odd-degree vertex), so the statement holds. Now assume we know it for cases up to k , and take a forest T with $2(k + 1)$ odd-degree vertices. Let P be a maximal path in T . We have seen in class that P will connect two leaves. We claim that if we delete the edges of P then we get a forest $T - P$ with $2k$ odd-degree vertices. Indeed, the two leaves will lose the edge touching them, so they have degree 0 in $T - P$, while every other vertex loses either 0 or 2 incident edges, hence the parity of its degree does not change. So we can apply induction on $T - P$ to get k paths partitioning the edge set of $T - P$. Together with path P we have $k + 1$ paths partitioning the edge set of T .