Graph Theory - Problem Set 1 (Solutions)

September 12, 2024

Exercises

1. Given a graph G with vertex set $V = \{v_1, \ldots, v_n\}$ we define the *degree sequence* of G to be the list $d(v_1), \ldots, d(v_n)$ of degrees in decreasing order. For each of the following lists, give an example of a graph with such a degree sequence or prove that no such graph exists:

   (a) $3, 3, 2, 2, 2, 1$

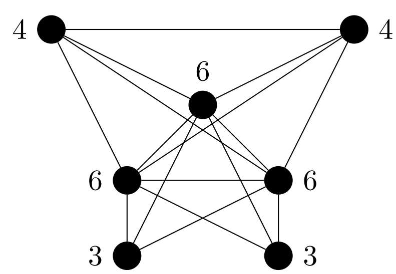
   (b) $6, 6, 6, 4, 4, 3, 3$

   (c) $6, 6, 6, 4, 4, 2, 2$

   Solution:

   (a) No. Since by Problem 5, the number of odd-degree vertices in a graph is always even.

   (b) Yes. Consider the following graph. The degree of each vertex is indicated next to it.

   (c) No. Since otherwise we have 3 vertices of degree 6 which are adjacent to all other vertices of the graph; so each vertex in the graph must be of degree at least 3.

2. Construct two graphs that have the same degree sequence but are not isomorphic.

   Solution: Let G_1 be of a cycle on 6 vertices, and let G_2 be the union of two disjoint cycles on 3 vertices each. In both graphs each vertex has degree 2, but the graphs are not isomorphic, since one is connected and the other is not.

3. A graph is *k-regular* if every vertex has degree k. How do 1-regular graphs look like? And 2-regular graphs?

   Solution: A 1-regular graph is just a disjoint union of edges (soon to be called a matching). A 2-regular graph is a disjoint union of cycles.

4. How many (labelled) graphs exist on a given set of n vertices? How many of them contain exactly m edges?

   Solution: Since there are $\binom{n}{2}$ possible edges on n vertices, and a graph may or may not have each of these edges, we get that there are $2^{\binom{n}{2}}$ possible graphs on n vertices.

   For the second problem, out of the $\binom{n}{2}$ possible edges, we want to choose m ones. So there are $\binom{\binom{n}{2}}{m}$ possible graphs on n vertices and with m edges.

Problems

5. Prove that the number of odd-degree vertices in a graph is always even.

   Solution: Let $G = (V, E)$ be an arbitrary graph. In the lecture we have proved that $\sum_{v \in V} d(v) = 2|E|$.

   Let $V_1 \subseteq V$ be the set of vertices of G which have odd degree and $V_2 = V \backslash V_1$ be the set of vertices of G which have even degree. We have that

   $$\sum_{v \in V} d(v) = \sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) = 2|E|.$$

   Since all the vertices in V_2 have even degree, and $2|E|$ is even, we obtain that $\sum_{v \in V_1} d(v)$ is even. But since V_1 is the set of vertices of odd degree, we obtain that the cardinality of V_1 is even (that is, there are an even number of vertices of odd degree), which completes the proof.

6. Let W be a closed walk that uses the edge e exactly once. Prove that W contains a cycle through e.

   Solution. Let $v_1 v_2 \ldots v_n v_1$ be a shortest closed walk that uses the edge e exactly once. We claim that this walk is a cycle. Indeed, if $v_i = v_j$ for some $i < j$, then either the closed walk $v_1 \ldots v_i v_{j+1} \ldots v_1$ or the closed walk $v_i v_{i+1} \ldots v_j$ uses the edge e exactly once, and both of them are shorter, which is not possible. (Why doesn't this argument work for an arbitrary walk that uses the edge e exactly once?)

7. Show that every graph on at least two vertices contains two vertices of equal degree.

   Solution: Suppose not. Then there is a graph of n vertices where all vertices have different degrees. Since the degree of a vertex is at most $n-1$, the set of degrees must be

   $$\{0, 1, 2, \ldots, n-2, n-1\}.$$

   However the vertex with degree $n-1$ has to be adjacent to all other vertices, while the one with degree 0 is not adjacent to any vertex, a contradiction.

8. What is the maximum number of edges in a bipartite graph on n vertices? (Prove your answer.)

   Solution: Let $G = (A \cup B, E)$ be a bipartite graph, with A, B disjoint and $|A| + |B| = n$. Since all the edges of G have one endpoint in A and the other in B, the number of edges $|E|$ of G cannot exceed the number of pairs $(a, b) \in A \times B$, so $|E| \leq |A| \cdot |B| = |A|(n - |A|)$. Intuitively, such a product is maximized when the two factors are equal, so when $|A| = \lfloor n/2 \rfloor$. More formally, we can use the inequality $4xy \leq (x+y)^2$ to get

   $$|E| \leq |A|(n - |A|) \leq \frac{(|A| + n - |A|)^2}{4} = \frac{n^2}{4}.$$

   Therefore, the number of edges of a bipartite graph on n edges is at most $n^2/4$.

   Note that $n^2/4$ is exactly the maximum when n is even, because then it is attained by the complete bipartite graph $K_{n/2,n/2}$. When n is odd, the maximum is actually $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil = \frac{n^2-1}{4}$, which is attained by $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

9. (*) Let G be a graph that contains a cycle C, and a path of length at least k between some two vertices of C (but they can also intersect elsewhere). Show that G contains a cycle of length at least $\sqrt{k}$.

   Solution: Let P be the u-v path where u, v are vertices in C. Let $V(P)$ and $V(C)$ be vertices of the path P and cycle C respectively. If $|V(P) \cap V(C)| \geq \sqrt{k}$, then C is already a cycle in G of length at least $\sqrt{k}$. Otherwise we have $|V(P) \cap V(C)| < \sqrt{k}$, then there exists two vertices $u', v' \in V(P) \cap V(C)$ such that there is a u'-v' subpath in P of length at least $\sqrt{k}$ and internally disjoint with the u'-v' subpath in C. Join these two subpaths in P and C together to get a cycle of length at least $\sqrt{k}$.