Graph Theory - Problem Set 12 (Solutions)

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Exercises

1. Using k colors, construct a coloring of the edges of the complete graph on 2^k vertices without creating a monochromatic triangle.

   Solution. We can construct the desired coloring by using induction on k. Suppose that we have constructed an edge coloring c of the complete graph on 2^{k-1} vertices with $k-1$ colors, say, $1, 2, \ldots, k-1$. Now take two vertex-disjoint copies of this graph (so we have taken a total of $2^{k-1} + 2^{k-1} = 2^k$ vertices). Color the edges inside each of the copies according to the coloring c (i.e., using only $k-1$ colors), and color all of the edges between the two copies using a new color k. It is easy to check that this gives the desired coloring with k colors, and it contains no monochromatic triangles.

2. The lower bound for $R(s,s)$ that we saw in the lecture is not a constructive proof: it merely shows the *existence* of a red-blue coloring not containing any monochromatic copy of K_s by bounding the number of bad graphs. Give an explicit coloring on $K_{(s-1)^2}$ that proves $R(s,s) > (s-1)^2$.

   Solution. Take $s-1$ disjoint K_{s-1}'s, color them blue, and color the complement red. This is a 2-coloring of $K_{(s-1)^2}$, and it has no blue clique of size s (two of the s vertices would belong to different K_{s-1}'s, so they would be connected in red), and it has no red clique of size s, either (two of the s vertices would belong to the same K_{s-1}, so they would be connected in blue).

3. A random graph $G(n,p)$ is a probability space of all labeled graphs on n vertices $\{1, 2, \ldots, n\}$, where for each pair $1 \leq i < j \leq n$, (i,j) is an edge of $G(n,p)$ with probability p, independently of any other edge (you can think of a sequence of independent coin tosses for each edge).

   (a) the expected number of edges in $G(n,p)$;
   (b) the expected degree of a vertex in $G(n,p)$;
   (c) the expected number of triangles (cycles of length 3) in $G(n,p)$;
   (d) the expected number of paths of length 2 in $G(n,p)$;
   (e) the probability that the degree of a given vertex v is exactly k.

   Solution.

   (a) The expected number of edges in $G(n,p)$ is $\binom{n}{2}p$.
   (b) The expected degree of a vertex in $G(n,p)$ is $(n-1)p$.
   (c) The expected number of triangles in $G(n,p)$ is $\binom{n}{3}p^3$. The number of possible triangles is $\binom{n}{3}$ and each of them arises with probability p^3.
   (d) The expected number of paths of length 2 in $G(n,p)$ is $3\binom{n}{3}p^2$. The number of possible paths of length 2 is $3\binom{n}{3}$ and each of them arises with probability p^2.

(e) The probability that the degree of a given vertex v is exactly k is $\binom{n-1}{k}p^k(1-p)^{(n-1-k)}$.

Problems

4. Prove that $R(n_1, \ldots, n_k) \leq R(n_1, \ldots, n_{k-2}, R(n_{k-1}, n_k))$. Deduce that for every k and n, there is an N such that any k-coloring of the edges of K_N contains a monochromatic K_n.

   Solution. Let $r = R(n_1, \ldots, n_{k-2}, R(n_{k-1}, n_k))$. We want to show that any k-edge-coloring of K_r will contain a clique of size n_i in color i for some $1 \leq i \leq k$. By the definition of r, either there is such an $i \in \{1, \ldots, k-2\}$, or there is a clique of size $R(n_{k-1}, n_k)$ that only uses colors $k-1$ and k. But then the definition of $R(n_{k-1}, n_k)$, there is either a clique of size n_{k-1} in color $k-1$ or a clique of size n_k in color k. This is what we wanted to show. Now induction on k shows that these multicolor Ramsey numbers are indeed finite (the $k = 2$ case was established in class).

5. Show that the edges of K_n can be colored with 3 colors so that the number of monochromatic triangles is at most $\frac{1}{9}\binom{n}{3}$.

   Solution. Let X be a random variable counting the number of monochromatic triangles in a random coloring of the edges of K_n with 3 colors, and let X_T be a random variable taking value 1 if a given triangle T is monochromatic, and 0 otherwise (a triangle is a triple of vertices of K_n). Since the total number of possible colorings of T is 3^3, and there are 3 ways to color T as a monochromatic triangle, we have that $\mathbb{E}[X_T] = 3/3^3$.

   Since $X = \sum_T X_T$, by the linearity of expectation (and since there are $\binom{n}{3}$ possible distinct triangles in K_n), we have that

   $$\mathbb{E}[X] = \mathbb{E}[\sum_T X_T] = \sum_T \mathbb{E}[X_T] = \binom{n}{3} \cdot \frac{1}{9}.$$

   Thus, there exists a coloring of K_n such that the number of monochromatic triangles is at most $\binom{n}{3}/9$.

6. (a) Show that if for some real number $0 \leq p \leq 1$ we have $\binom{n}{s}p^{\binom{s}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$, then $R(s,t) > n$.

   (b) Deduce that there is a positive constant c such that $R(4,t) \geq c \cdot \frac{t^{3/2}}{\log^{3/2} t}$.

   Solution.

   (a) Consider the random coloring of the edges of K_n by red and blue, such that each edge is colored independently by red with probability p, and by blue with probability $1-p$. Clearly, the expected number of monochromatic cliques of size s in $G(n,p)$ is $\binom{n}{s}p^{\binom{s}{2}}$. Therefore, the sum of expected numbers of red K_s's and blue K_t's is $m := \binom{n}{s}p^{\binom{s}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}}$. Since $m < 1$ by the assumption, there exists a coloring χ without any red K_s or blue K_t. Therefore, we have $R(s,t) > n$.

   (b) We want to make $\binom{n}{4}p^{\binom{4}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}}$ less than 1 for large n. $\binom{n}{4} \leq n^4/2$, so for $p = n^{-2/3}$ we have $\binom{n}{4}p^{\binom{4}{2}} < 1/2$. For this p the second term is

   $$\binom{n}{t}(1-p)^{\binom{t}{2}} \leq n^t e^{-p\binom{t}{2}} \leq \exp(t \log n - n^{-2/3}t^2/4)$$

If $n \le c\frac{t^{3/2}}{\log^{3/2} t}$ for some small enough c then $n^{-2/3}t^2/4 > t\log n + 1$, hence this term is also less than $1/2$ and we can apply part (a).

7. Prove that for every $k \ge 2$ there exists an integer N such that every coloring of $[N] = \{1, \dots, N\}$ with k colors contains three numbers a, b, c satisfying $ab = c$ that have the same color.

   Solution. According to Schur's theorem, there is a K such that every coloring of $[K]$ with k colors contains three numbers x, y, z satisfying $x + y = z$. Now let $N = 2^K$ and take an arbitrary k-coloring c of $[N]$. Let d be a coloring of $[K]$ defined by $d(i) = c(2^i)$. By Schur's theorem, there are x, y, z such that $x + y = z$ and $d(x) = d(y) = d(z)$. But then $2^x \cdot 2^y = 2^{x+y} = 2^z$ and $d(x) = c(2^x) = c(2^y) = c(2^z)$, which is what we wanted.

8. (a) Prove that $R(4, 3) \le 10$, i.e., any graph on 10 vertices contains a clique of size 4 or an independent set of size 3.

   Solution. Take an arbitrary vertex v. It either has 6 neighbors in red or 4 neighors in blue. In the former case, those six neighbors induce a monochromatic triangle ($R(3, 3) = 6$). If it's blue, we are done, if it's red then we get a red K_4 with v. In the latter case, if the 4 blue neighbors induce a blue edge, we get a blue triangle, otherwise we get a red K_4.

   (b) Prove that $R(4, 3) \le 9$.

   Solution. Take an arbitrary vertex v. If it has 6 neighbors in red or 4 neighbors in blue then we are done as before. So we may assume that it has 5 red neighbors and 3 blue neighbors. In fact, we may assume this for every vertex v. But then the red graph is a 5-regular graph on 9 vertices, which is impossible because the sum of the degrees is always even. This contradiction shows that for some v, we can indeed repeat the argument from part (a).