Graph Theory - Problem Set 11 (Solutions)

November 28, 2024

Exercises

1. Let $(\Omega, \mathbb{P})$ be a probability space. Prove that for any collection of events $\mathcal{E}_1, \ldots, \mathcal{E}_k$, we have

$$\mathbb{P}\left[\bigcup_{i=1}^{k} \mathcal{E}_i\right] \leq \sum_{i=1}^{k} \mathbb{P}[\mathcal{E}_i],$$

and if $\mathcal{E}_1, \ldots, \mathcal{E}_k$ are disjoint events, then we have equality here.

Solution. The proof is by induction on n. For two events $\mathcal{E}_1$ and $\mathcal{E}_2$ we have that

$$\mathbb{P}[\mathcal{E}_1 \cup \mathcal{E}_2] = \sum_{\omega \in \mathcal{E}_1 \cup \mathcal{E}_2} p(\omega) \leq \sum_{\omega \in \mathcal{E}_1} p(\omega) + \sum_{\omega \in \mathcal{E}_2} p(\omega) = \mathbb{P}[\mathcal{E}_1] + \mathbb{P}[\mathcal{E}_2].$$

Assume that the statement holds for $n-1$ and let us prove it for n:

$$\mathbb{P}\left[\bigcup_{1}^{n} \mathcal{E}_i\right] = \mathbb{P}\left[\left(\bigcup_{1}^{n-1} \mathcal{E}_i\right) \cup \mathcal{E}_n\right] \leqslant \mathbb{P}\left[\bigcup_{1}^{n-1} \mathcal{E}_i\right] + \mathbb{P}[\mathcal{E}_n] \leqslant \sum_{1}^{n-1} \mathbb{P}[\mathcal{E}_i] + \mathbb{P}[\mathcal{E}_n] = \sum_{1}^{n} \mathbb{P}[\mathcal{E}_i].$$

2. Let σ be an arbitrary permutation of $\{1, \ldots, n\}$, selected uniformly at random from the set of all permutations, that is, each permutation is selected with probability $\frac{1}{n!}$. Recall that i is a fixed point if $\sigma(i) = i$. What is the expectation of the number of fixed points in σ?

Solution. Let X_i be a random variable being 1 if the i-th position is a fixed point, and 0 otherwise. Then, by linearity of expectation, the expected number of fixed points in a random permutation are simply $\sum_{i=1}^{n} \mathbb{E}[X_i]$. On the other hand, $\mathbb{E}[X_i] = 1/n$, so we obtain that

$$\sum_{i=1}^{n} \mathbb{E}[X_i] = n \cdot 1/n = 1.$$

3. Take a complete graph K_n where each edge is independently colored red, green or blue with probability 1/3. What is the expected number of red cliques of size a in this graph?

Solution. We assume that vertices of G are labeled from 1 to n. Let X be the number of crossing edges. We decompose

$$X = \sum_{I \in \binom{[n]}{a}} X_I,$$

where $\binom{[n]}{a}$ is the set of a-element subsets of the set $\{1, \ldots, n\}$ and X_I is the indicator random variable for clique on vertices with indices from I being red colored.

By linearity of expectation, we have

$$\mathbb{E}[X] = \sum_{I \in \binom{[n]}{a}} \mathbb{E}[X_I].$$

On the other hand $\mathbb{E}[X_I] = \left(\frac{1}{3}\right)^{\binom{a}{2}}$, and, finally, the expected number of red cliques of size a in this graph is

$$\mathbb{E}[X] = \binom{n}{a}\left(\frac{1}{3}\right)^{\binom{a}{2}}$$

4. Prove that $\alpha(G) \geq \frac{n^2}{2m+n}$ for every graph with n vertices and m edges follows from Turán's theorem (in fact, they are essentially equivalent).

   Solution. Let $\alpha = \alpha(G)$ and let H be the complement of G. Then H contains no $K_{\alpha+1}$, so by Turán's theorem, H has at most $(1-\frac{1}{\alpha})\frac{n^2}{2}$ edges. But then G has $m \geq \frac{n(n-1)}{2} - (\frac{n^2}{2} - \frac{n^2}{2\alpha}) = \frac{n^2}{2\alpha} - \frac{n}{2}$ edges. Rearranging this, we get $2m + n \geq \frac{n^2}{\alpha}$ and hence $\alpha \geq \frac{n^2}{2m+n}$.

5. In this exercise, we prove the following two results which are used in the proof of Erdös theorem (existence of a graph with large girth and large chromatic number).

   (a) The expectation of the number of ℓ-cycles, $3 \leq \ell \leq n$, in $G \in \mathcal{G}(n,p)$ is: $\frac{n(n-1)\ldots(n-\ell+1)}{2\ell}p^\ell$.

   (b) For any integers n and k such that $n \geq k \geq 2$, the probability that a graph $G \in \mathcal{G}(n,p)$ has an independent set larger than k is at most: $\Pr[\alpha(G) \geq k] \leq \binom{n}{k}(1-p)^{\binom{k}{2}}$.

   Solution.

   (a) Let $X : \mathcal{G}(n,p) \to \mathbb{N}$ be the random variable that assigns to a random graph G its number of ℓ-cycles. Let $\mathcal{C}_\ell$ be the set of all ℓ-cycles in the complete graph with same vertices as any G. Since there are $n(n-1)\ldots(n-\ell+1)$ ways of choosing a sequence of ℓ distinct vertices, and each ℓ-cycle is identified by 2ℓ of those sequences, we have

   $$|\mathcal{C}_\ell| = \frac{n(n-1)\ldots(n-\ell+1)}{2\ell}p^\ell.$$

   For each $C \in \mathcal{C}_\ell$, define $X_C : \mathcal{G}(n,p) \to \{0,1\}$ to be the indicator random variable for $C \subseteq G$. We have $\mathbb{E}[X_C] = \mathbb{P}[C \subseteq G] = p^\ell$. Since $X(G) = \sum_{C\in\mathcal{C}_\ell} X_C(G)$, the expectation of X follows from the linearity of expectation.

   (b) The probability that a fixed k-set is independent in G is $(1-p)^{\binom{k}{2}}$. The assertion then follows from the fact that there are only $\binom{n}{k}$ such k-sets.

Problems

6. Let G be a graph with m edges, and let $X \subseteq V(G)$ be a random set that contains each vertex of G independently with probability $1/2$. Let $G[X]$ be the induced subgraph of G with vertex set X and contains all edges in G with both ends in X. What is the expected number of edges in $G[X]$?

   Solution. Let Y be the number of edges in $G[X]$. Then $Y = \sum_{e\in E(G)} Y_e$, where Y_e is the indicator of the event that the edge e is induced by X. This event occurs if and only if both endpoints of e are included in X, which has probability $1/4$. So by the linearity of expectation, $\mathbb{E}[Y] = \sum_{e\in E(G)} \mathbb{E}[Y_e] = m/4$.

7. Let G be a graph with m edges, and let k be a positive integer. Prove that the vertices of G can be colored with k colors in such a way that there are at most m/k monochromatic edges (i.e., edges with both endpoints colored the same).

   Solution. We assume that edges of G are labeled from 1 to m. For any $1 \leqslant i \leqslant m$, we define the random variable X_i being 1 whenever endpoints of the i-th edge are colored by the same color, and 0 otherwise. Let $X = \sum_i^m X_i$. Then $\mathbb{E}[X] = \sum_i^m \mathbb{E}[X_i]$. On the other hand, it is easy to see that $\mathbb{E}[X_i] = \frac{k}{k^2} = \frac{1}{k}$. Therefore, $\mathbb{E}[X] = \frac{m}{k}$. Thus, there is a coloring of the vertices of G with k colors such that at most $\frac{m}{k}$ edges of G connect two vertices with the same color.

8. Prove that if G has $2n$ vertices and e edges then it contains a bipartite subgraph with at least $e \cdot \frac{n}{2n-1}$ edges.

   Hint: Use a random partition of the vertices into two parts of size n.

   Solution. The proof is similar to the probabilistic proof for the bound $e/2$ given in lectures, but now we choose a more subtle probability space. Let A be an n-element subset of $V(G)$ chosen uniformly among all n-element subsets of $V(G)$. Set $B = V(G) - T$. Call an edge $\{x, y\}$ crossing if exactly one of x, y are in A. Then any edge $\{x, y\}$ now has probability $\frac{n}{2n-1}$ of being crossing.

   We complete the proof with the standard arguments. Let X be the number of crossing edges and X_{xy} be the indicator random variable for $\{x, y\}$ being crossing. By linearity of expectation,

$$\mathbb{E}[X] = \sum_{\{x,y\} \in E(G)} \mathbb{E}[X_{xy}] = e\frac{n}{2n-1}.$$

   Thus, there is a choice of T such that $X \geqslant e\frac{n}{2n-1}$ and the set of those crossing edges form a bipartite graph.

9. Prove that $\mathrm{ex}(n, C_{2k}) > \frac{1}{16}n^{1+1/(2k-1)}$ for every $n, k \geq 2$.

   Hint: Apply the same idea as in proving a lower bound of $\mathrm{ex}(n, K_{s,t})$.

   Solution. This proof follows the same idea used in the lectures for $K_{s,s}$. Let $p = \frac{1}{2}n^{-1+1/(2k-1)}$, and consider the random graph G on n vertices that contains each edge independently with probability p. Let X be the number of edges in G, then $\mathbb{E}[X] = \frac{n(n-1)p}{2} \geq \frac{1}{8}n^{1+1/(2k-1)}$. Let Y be the number of cycles of length $2k$ in G. There are at most $n(n-1)\dots(n-2k+1) \leq n^{2k}$ potential cycles, and each occurs in G with probability p^{2k}, so

$$\mathbb{E}[Y] \leq n^{2k}p^{2k} = \frac{1}{2^{2k}}n^{2k/(2k-1)} \leq \frac{1}{16}n^{1+1/(2k-1)}$$

   Now let H be a subgraph of G obtained by deleting an edge from each $2k$-cycle of G. Then in expectation, H has at least $\mathbb{E}[X - Y] = \mathbb{E}[X] - \mathbb{E}[Y] \geq \frac{1}{16}n^{1+1/(2k-1)}$ edges, so there is an instance with at least this many edges. This graph is C_{2k}-free.

   Remark: one can get better constants for large n by choosing p more carefully.