

Graph Theory - Problem Set 10 (Solutions)

November 21, 2024

Exercises

1. Prove that if G is a K_3 -free graph, then $\alpha(G) \geq \Delta(G)$.

Solution. If G is triangle-free then every neighborhood is an independent set, so $\alpha(G) \geq \Delta(G)$.

2. Prove the lower bound for the Erdős-Stone-Simonovits theorem, i.e., for every graph H with chromatic number $s \geq 2$, $\text{ex}(n, H) \geq |E(T(n, s-1))|$.

Solution. The Turán graph $T(n, s-1)$ is $(s-1)$ -colorable, so it does not contain any subgraph of chromatic number s , in particular, $\text{ex}(n, H) \geq |E(T(n, s-1))|$.

3. Deduce from the proof of Mantel's theorem that $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the only "extremal" K_3 -free graph, i.e., every K_3 -free graph with $\text{ex}(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$ edges is isomorphic to G .

Solution. Analyzing the proof, we see that the optimum is reached when $\Delta = \lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$, and every vertex not adjacent to v touches exactly Δ edges, which are all distinct. This is only possible if there is no edge going between two non-neighbors of v , i.e., the non-neighbors are all connected to the neighbors of v . Then the graph is isomorphic to $K_{\Delta, n-\Delta}$.

4. Let L be a set of n lines in the plane and P a set of n points in the plane. Prove that the number of point-line incidences, i.e., pairs $(p, \ell) \in P \times L$ with $p \in \ell$ is $O(n^{3/2})$.

Solution. Let G be the bipartite graph on $2n$ vertices with parts P and L as vertices, where a point in P is connected to a line in L if the point lies on the line. Note that this graph is $K_{2,2}$ -free. Indeed, two lines in L share at most 1 point in common. Then we can apply the Kővári-Sós-Turán theorem on G to see that the graph has at most $cn^{3/2}$ edges for some constant c .

Problems

5. Recall from Problem Set 8 that $\alpha(G) + \tau(G) = |V(G)|$. Prove that if G is triangle-free then $|E(G)| \leq \alpha(G) \cdot \tau(G)$, and use this to reprove Mantel's theorem.

Solution. By Exercise 1, we have $\alpha(G) \geq \Delta(G)$. Now take a vertex cover of size $\tau(G)$. They touch all edges. On the other hand, the number of edges touching these vertices is at most $\Delta(G) \cdot \tau(G) \leq \alpha(G) \cdot \tau(G)$, proving the first statement. The second statement is just Cauchy-Schwarz (or AM-GM): if $x + y = n$ then $xy \leq n^2/4$.

6. Prove the Kővári-Sós-Turán theorem: For any integers $2 \leq s \leq t$ there is a constant c such that $\text{ex}(n, K_{s,t}) \leq cn^{2-1/s}$.

Hint: Use a similar double-counting argument as in the proof of $\text{ex}(n, K_{2,2}) \leq O(n^{3/2})$.

Solution. The proof follows the $K_{2,2}$ case seen in the lectures. We give a sketch here, without explaining the calculations in detail.

Let G be a $K_{s,t}$ -free graph on n vertices and m edges. The idea is to count the number of pairs $(v, \{x_1, \dots, x_s\})$, where $v, x_1, \dots, x_s$ are vertices, and v is adjacent to all of $x_1, \dots, x_s$. On the one hand, this is at least

$$\sum_{v \in V} \binom{d(v)}{s} \geq n \cdot \binom{\frac{\sum d(v)}{n}}{s} = n \cdot \binom{2m/n}{s} > Cn(m/n)^s$$

for some constant C . On the other hand, every s -tuple has at most $t - 1$ common neighbors, so this number is at most

$$(t - 1) \binom{n}{s} \leq Dn^s$$

for some constant D . But then $Cn(m/n)^s \leq Dn^s$, which is equivalent to $m/n \leq ((D/C)n^{s-1})^{1/s}$, and thus $m \leq (D/C)^{1/s} \cdot n^{2-1/s}$, as needed.

7. Let G be a d -regular graph on n vertices with girth at least $2k + 1$. Prove that $d \leq n^{1/k}$, i.e., G has at most $\frac{1}{2}n^{1+1/k}$ edges.

Solution. Suppose $d > n^{1/k}$. Let v be a vertex and let $V_1, \dots, V_k$ be subsets of vertices where V_i is the set of vertices of distance i from v . As there is no cycle of length less than $2k + 1$, no vertex in V_i is connected to multiple vertices in V_{i-1} . The regularity then implies $|V_i| = d(d - 1)^{i-1}$. But then G contains at least $|V_k| \geq d(d - 1)^{k-1} > (d - 1)^k \geq n$, a contradiction.

8. Show that $\text{ex}(n, \triangleright) = \lfloor \frac{n^2}{4} \rfloor$ for every $n > 3$.

Hint: Modify slightly the proof of Mantel's theorem.

Solution. Take a vertex v of max degree Δ . If $\Delta \geq 3$ then the neighborhood of v contains no edge: otherwise we get a graph which is forbidden. We can then repeat the proof of Mantel to get that the number of edges is at most $\Delta(n - \Delta) \leq \lfloor \frac{n^2}{4} \rfloor$. Otherwise, $\Delta \leq 2$, so the total number of edges is at most $n\Delta/2 \leq n$. When $n > 3$, this is at most $\lfloor \frac{n^2}{4} \rfloor$, so we are done.

Another proof is to use Mantel's theorem: We know there is a triangle in any graph with more than $\lfloor \frac{n^2}{4} \rfloor$ edges. This forms a graph in question, unless this triangle is isolated. One can then delete the vertices of the triangle and iterate on the remaining, denser graph. Eventually this will yield a graph that is way too dense not to contain the graph. (fyi: this was just a solution sketch).

9. This exercise is about constructing a $K_{2,2}$ -free graph on n vertices with $n^{3/2}$ edges for large n .

Let $p \geq 3$ be a prime, and G_0 be the graph on the vertex set $\mathbb{Z}_p \times \mathbb{Z}_p$ where (x, y) and (x_1, y_1) are connected by an edge and only if $x + x_1 = yy_1$. (Technically this is a multigraph as it has loops.) Let G be the graph on $n = p^2$ vertices that we get by deleting the loops from G_0 .

- (a) Prove that G_0 is p -regular and has at most p loops.
- (b) Deduce that G has $(\frac{1}{2} + o(1))n^{3/2}$ edges.
- (c) Show that any two vertices in G have at most 1 common neighbor (and hence G is $K_{2,2}$ -free).

Solution.

- (a) For every x, y, y_1 there is a unique choice of x_1 such that (x, y) and (x_1, y_1) are adjacent. Also, loops correspond to solutions of the equation $2x = y^2$. There is therefore one loop for every choice of y , giving at most p loops in total.
- (b) G has $n = p^2$ vertices, and $\frac{1}{2}(np - p) = (\frac{1}{2} + o(1))n^{3/2}$ edges.
- (c) For any (x_1, y_1) and (x_2, y_2) , a vertex (x, y) adjacent to both of them satisfies $x + x_1 = yy_1$ and $x + x_2 = yy_2$, so $x_1 - x_2 = y(y_1 - y_2)$. If $y_1 = y_2$ then $x_1 = x_2$, so if our chosen vertices (x_1, y_1) and (x_2, y_2) were distinct then $y_1 - y_2 \neq 0$. Then y is uniquely determined from the last equation, and this defines x , as well.