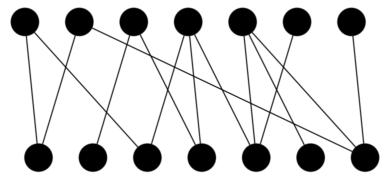
Graph Theory - Problem Set 8

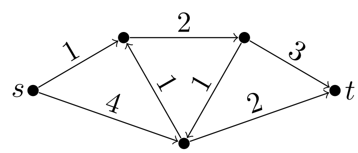
November 7, 2024

Exercises

1. Find a minimum vertex cover in the following graph.



2. Find a maximum flow from s to t and a minimum s-t cut in the following network.



3. Construct a network on four vertices for which the Ford-Fulkerson algorithm may need more than a million iterations, depending on the choice of augmenting paths.

4. Let G be a network with source s, sink t, and integer capacities. Prove or disprove the following statements:

   (a) If all capacities are even then there is a maximal flow f such that $f(e)$ is even for all edges e.

   (b) If all capacities are odd then there is a maximal flow f such that $f(e)$ is odd for all edges e.

Problems

5. Let G be a graph on n vertices

   (a) Show that $\tau(G) + \alpha(G) = n$.

   (b) Show that if G is bipartite then $\nu(G) + \alpha(G) = n$.

6. Let G be a network with source s, sink t, and integer capacities. Prove that an edge e is saturated (i.e., the flow uses its full capacity) in every maximum s-t flow if and only if decreasing the capacity of e by 1 would decrease the maximum value of an s-t flow in G.

7. Deduce Hall's theorem from the max-flow min-cut theorem.

   Hint: Consider a network derived from the bipartite graph similar to the proof of König theorem.

8. Let A be an $n \times m$ matrix of non-negative real numbers such that the sum of the entries is an integer in every row and in every column. Prove that there is an $n \times m$ matrix B of non-negative integers with the same sums as in A, in every row and every column.