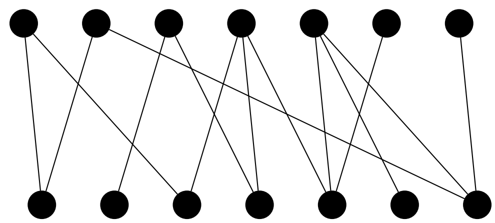
Graph Theory - Problem Set 7

October 31, 2024

Exercises

1. Construct preference lists for the vertices of $K_{3,3}$ so that there are multiple stable matchings.
2. Find a maximum matching in the following graph.



3. Construct a 2-regular graph without a perfect matching.
4. Let G be a bipartite graph on $2n$ vertices such that $\alpha(G) = n$.
   (a) Show that both parts of G contain n vertices.
   (b) Check that Hall's condition holds for G and then deduce that G has a perfect matching.

Problems

5. Prove the following "defect" version of Hall's theorem:

   If $G = (A \cup B, E)$ is a bipartite graph such that $|N(S)| \geq |S| - d$ holds for every $S \subseteq A$, then G has a matching with at least $|A| - d$ edges.

6. An $r \times s$ *Latin rectangle* is an $r \times s$ matrix A with entries in $\{1, \ldots, s\}$ such that each integer occurs at most once in each row and at most once in each column. An $s \times s$ Latin rectangle is called a *Latin square*. Prove that every $r \times s$ Latin rectangle can be extended to an $s \times s$ Latin square.

   Hint: Consider a bipartite graph that models the constraints of any $i \in \{1, \ldots, s\}$ appearing only once in each row and column.

7. Let G be a bipartite graph with both parts of the same size $2n$ and minimum degree at least n. Prove that G has a perfect matching.

8. Prove König's line coloring theorem: for every bipartite graph G, we have $\chi'(G) = \Delta(G)$.

   Hint: One proof is very similar (while simpler) to the proof of Vizing's theorem.

9. Prove that every bipartite graph G has a matching of size at least $|E(G)|/\Delta(G)$.