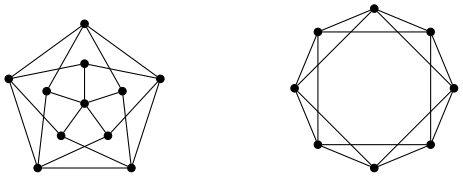
Graph Theory - Problem Set 6

October 17, 2024

Exercises

1. Determine the chromatic number of the first graph and the edge-chromatic number of the second graph below.



2. For a graph G, we define $G[X]$, the subgraph *induced* by the vertex set $X \subseteq V(G)$ as the graph with vertex set X that contains all the edges of G with both ends in X.
Prove that $\chi(G) \leq \chi(G[X]) + \chi(G[V \setminus X])$.

3. Are the following statements true? Provide reasons for your answers.

   (a) If G and H are graphs on the same vertex set, then $\chi(G \cup H) \leq \chi(G) + \chi(H)$.

   (b) Every graph G has a coloring with $\chi(G)$ colors where $\alpha(G)$ vertices get the same color.

Problems

4. Let G be a graph on n vertices and $\overline{G}$ be its complement. Prove that

   (a) $\chi(G)\chi(\overline{G}) \geq n$.

   (b) $\chi(G) + \chi(\overline{G}) \leq n + 1$. *Hint: Prove by induction on n.*

5. (a) Show that if an n-vertex graph is d-degenerate, then it has at most dn edges.

   (b) Prove that if the longest path in G has length ℓ, then $\chi(G) \leq \ell + 1$.

6. Let G be a 3-regular graph with $\chi'(G) = 4$. Prove that G does not have a Hamilton cycle.
Hint: What can be said about the parity of the number of vertices?

7. Prove that if every two odd cycles of G intersect in at least one vertex, then $\chi(G) \leq 5$.
Hint: Consider two cases in which G contains odd cycles or not.

8. (a) Find the edge-chromatic number of K_{2n+1}.

   (b) Find the edge-chromatic number of K_{2n}.