Graph Theory - Problem Set 4

October 3, 2024

Exercises

1. In this exercise, we show that the sufficient conditions for Hamiltonicity that we saw in the lecture are tight in some sense.

   (a) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has $\binom{n-1}{2} + 1$ edges.

   (b) For every $n \geq 2$, find a non-Hamiltonian graph on n vertices that has minimum degree $\lceil \frac{n}{2} \rceil - 1$.

   (c) Find a non-Hamiltonian graph G which satisfies $\alpha(G) = \kappa(G) + 1$.

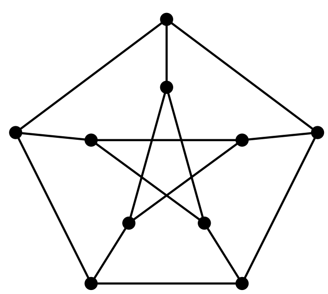
2. For every $k, n \geq 2$, find a graph G on *at least* n vertices such that $\delta(G) = k$ but G contains no cycle longer than $k + 1$.

3. Check that the proof of Dirac's theorem also proves Ore's theorem.

   <u>Dirac's theorem</u>: Let G be a graph on $n \geq 3$ vertices. If $\delta(G) \geq \frac{n}{2}$, then G contains a Hamilton cycle.

   <u>Ore's theorem</u>: Let G be a graph on $n \geq 3$ vertices. If $d(u) + d(v) \geq n$ for any non-adjacent vertices u and v, then G contains a Hamilton cycle.

4. The graph below is called the Petersen graph. Does it have a Hamilton path? Does it have a Hamilton cycle? Provide reasons for your answers.



Problems

5. Use Ore's theorem from Exercise 3 to give a short proof of the fact that any n-vertex graph G with more than $\binom{n-1}{2} + 1$ edges has a Hamilton cycle.

6. Let G be a connected graph on n vertices with minimum degree δ. Show that

   (a) if $\delta \leq \frac{n-1}{2}$ then G contains a path of length 2δ, and

   (b) if $\delta \geq \frac{n-1}{2}$ then G contains a Hamilton path.

7. Suppose that each edge of the complete graph K_n is painted either red or blue. Prove that this colored graph has a Hamilton path, which is the union of a red path and a blue path. (We allow the case when one of the paths has length 0, i.e., the Hamilton path uses only one color.)

8. Prove the following sufficient conditions of Hamiltonicity, which generalizes Dirac's theorem.

   Chvátal's theorem (1972): Let G be a graph on $n \geq 3$ vertices, whose degree sequence of its vertices is $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no $1 \leq k < \frac{n}{2}$ for which $d_k \leq k$ and $d_{n-k} < n-k$, then G is Hamiltonian.

   Hint: Show that its closure $c(G)$ is complete by contradiction. Start by taking two non-adjacent vertices in $c(G)$ with the largest sum of degrees.

9. (*) Let G be a graph in which every vertex has odd degree. Show that every edge of G is contained in an even number of Hamilton cycles.

   Hint: Let $xy \in E(G)$ be given. The Hamilton cycles through xy correspond to the Hamilton paths in $G - xy$ from x to y. Consider the set $\mathcal{H}$ of all Hamilton paths in $G - xy$ starting at x, and show that an even number of these end in y. To show this, define a graph on $\mathcal{H}$ so that the desired assertion follows from the fact (proved in Problem Set 1): the number of odd-degree vertices in a graph is always even.