Graph Theory - Problem Set 2

September 19, 2024

Exercises

1. Prove the triangle-inequality in graphs: for any three vertices u, v, w in a graph G,

$$d(u, v) + d(v, w) \geq d(u, w).$$

2. A graph that does not contain any cycles is called a *forest*.
   Prove that a forest on n vertices with c connected components has exactly $n - c$ edges.

3. Let T be a tree and e be an edge of T. Prove that $T - e$ is not connected.

4. Let T be a tree and let u and v be two non-adjacent vertices of T. Prove that $T + uv$ contains a unique cycle.

Problems

5. Let G be a graph with minimum degree $\delta > 1$. Prove that G contains a cycle of length at least $\delta + 1$.

6. Show that every tree T has at least $\Delta(T)$ leaves. (Where $\Delta(T)$ is the maximum degree of T.)

7. Let G be a graph on n vertices. Prove that if G has at least $2n - 1$ edges, then it contains an even cycle.

8. Let T be an n-vertex tree that has exactly $2k$ vertices of odd degree. Show that T can be split into k edge-disjoint paths, i.e., T is the union of k edge-disjoint paths. Note that we only consider path of length at least 1.

   Hint: Prove by induction.