Graph Theory - Problem Set 12

December 5, 2024

Exercises

1. Using k colors, construct a coloring of the edges of the complete graph on 2^k vertices without creating a monochromatic triangle.

2. The lower bound for $R(s,s)$ that we saw in the lecture is not a constructive proof: it merely shows the *existence* of a red-blue coloring not containing any monochromatic copy of K_s by bounding the number of bad graphs. Give an explicit coloring on $K_{(s-1)^2}$ that proves $R(s,s) > (s-1)^2$.

3. A random graph $G(n,p)$ is a probability space of all labeled graphs on n vertices $\{1, 2, \dots, n\}$, where for each pair $1 \le i < j \le n$, (i,j) is an edge of $G(n,p)$ with probability p, independently of any other edge (you can think of a sequence of independent coin tosses for each edge). Compute the following:

   (a) the expected number of edges in $G(n,p)$;

   (b) the expected degree of a vertex in $G(n,p)$;

   (c) the expected number of triangles (cycles of length 3) in $G(n,p)$;

   (d) the expected number of paths of length 2 in $G(n,p)$;

   (e) the probability that the degree of a given vertex v is exactly k.

Problems

4. Prove that $R(n_1, \dots, n_k) \le R(n_1, \dots, n_{k-2}, R(n_{k-1}, n_k))$. Deduce that for every k and n, there is an N such that any k-coloring of the edges of K_N contains a monochromatic K_n.

5. Show that the edges of K_n can be colored with 3 colors so that the number of monochromatic triangles is at most $\frac{1}{9}\binom{n}{3}$.

6. (a) Show that if for some real number $0 \le p \le 1$ we have $\binom{n}{s}p^{\binom{s}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$, then $R(s,t) > n$.

   (b) Deduce that there is a positive constant c such that $R(4,t) \ge c \cdot \frac{t^{3/2}}{\log^{3/2} t}$.

   Hint: Use $p = n^{-2/3}$ in (a) to deduce (b).

7. Prove that for every $k \ge 2$ there exists an integer N such that every coloring of $[N] = \{1, \dots, N\}$ with k colors contains three numbers a, b, c satisfying $ab = c$ that have the same color.

8. (a) Prove that $R(4,3) \le 10$, i.e., any graph on 10 vertices contains a clique of size 4 or an independent set of size 3.

   (b) Prove that $R(4,3) \le 9$.