Graph Theory - Problem Set 11

November 28, 2024

Exercises

1. Let $(\Omega, \mathbb{P})$ be a probability space. Prove that for any collection of events $\mathcal{E}_1, \ldots, \mathcal{E}_k$, we have
$$\mathbb{P}\left[\bigcup_{i=1}^{k} \mathcal{E}_i\right] \leq \sum_{i=1}^{k} \mathbb{P}[\mathcal{E}_i],$$
and if $\mathcal{E}_1, \ldots, \mathcal{E}_k$ are disjoint events, then we have equality here.

2. Let σ be an arbitrary permutation of $\{1, \ldots, n\}$, selected uniformly at random from the set of all permutations, that is, each permutation is selected with probability $\frac{1}{n!}$. Recall that i is a fixed point if $\sigma(i) = i$. What is the expectation of the number of fixed points in σ?

3. Take a complete graph K_n where each edge is independently colored red, green or blue with probability 1/3. What is the expected number of red cliques of size a in this graph?

4. Prove that $\alpha(G) \geq \frac{n^2}{2m+n}$ for every graph with n vertices and m edges follows from Turán's theorem (in fact, they are essentially equivalent).

5. In this exercise, we prove the following two results which are used in the proof of Erdös theorem (existence of a graph with large girth and large chromatic number).
   (a) The expectation of the number of ℓ-cycles, $3 \leq \ell \leq n$, in $G \in \mathcal{G}(n,p)$ is: $\frac{n(n-1)\ldots(n-\ell+1)}{2\ell}p^\ell$.
   (b) For any integers n and k such that $n \geq k \geq 2$, the probability that a graph $G \in \mathcal{G}(n,p)$ has an independent set larger than k is at most: $\Pr[\alpha(G) \geq k] \leq \binom{n}{k}(1-p)^{\binom{k}{2}}$.

Problems

6. Let G be a graph with m edges, and let $X \subseteq V(G)$ be a random set that contains each vertex of G independently with probability 1/2. Let $G[X]$ be the induced subgraph of G with vertex set X and contains all edges in G with both ends in X. What is the expected number of edges in $G[X]$?

7. Let G be a graph with m edges, and let k be a positive integer. Prove that the vertices of G can be colored with k colors in such a way that there are at most m/k monochromatic edges (i.e., edges with both endpoints colored the same).

8. Prove that if G has $2n$ vertices and e edges then it contains a bipartite subgraph with at least $e \cdot \frac{n}{2n-1}$ edges.
   Hint: Use a random partition of the vertices into two parts of size n.

9. Prove that $\mathrm{ex}(n, C_{2k}) > \frac{1}{16}n^{1+1/(2k-1)}$ for every $n, k \geq 2$.
   Hint: Apply the same idea as in proving a lower bound of $\mathrm{ex}(n, K_{s,t})$.