Graph Theory - Problem Set 10

November 21, 2024

Exercises

1. Prove that if G is a K_3-free graph, then $\alpha(G) \geq \Delta(G)$.

2. Prove the lower bound for the Erdős-Stone-Simonovits theorem, i.e., for every graph H with chromatic number $s \geq 2$, $\mathrm{ex}(n, H) \geq |E(T(n, s-1))|$.

3. Deduce from the proof of Mantel's theorem that $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the only "extremal" K_3-free graph, i.e., every K_3-free graph with $\mathrm{ex}(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$ edges is isomorphic to G.

4. Let L be a set of n lines in the plane and P a set of n points in the plane. Prove that the number of point-line incidences, i.e., pairs $(p, \ell) \in P \times L$ with $p \in \ell$ is $O(n^{3/2})$.

Problems

5. Recall from Problem Set 8 that $\alpha(G) + \tau(G) = |V(G)|$. Prove that if G is triangle-free then $|E(G)| \leq \alpha(G) \cdot \tau(G)$, and use this to reprove Mantel's theorem.

6. Prove the Kővári-Sós-Turán theorem: For any integers $2 \leq s \leq t$ there is a constant c such that $\mathrm{ex}(n, K_{s,t}) \leq cn^{2-1/s}$.

   Hint: Use a similar double-counting argument as in the proof of $\mathrm{ex}(n, K_{2,2}) \leq O(n^{3/2})$.

7. Let G be a d-regular graph on n vertices with girth at least $2k+1$. Prove that $d \leq n^{1/k}$, i.e., G has at most $\frac{1}{2}n^{1+1/k}$ edges.

8. Show that $\mathrm{ex}(n,$  $) = \lfloor \frac{n^2}{4} \rfloor$ for every $n > 3$.

   Hint: Modify slightly the proof of Mantel's theorem.

9. This exercise is about constructing a $K_{2,2}$-free graph on n vertices with $n^{3/2}$ edges for large n.

   Let $p \geq 3$ be a prime, and G_0 be the graph on the vertex set $\mathbb{Z}_p \times \mathbb{Z}_p$ where (x, y) and (x_1, y_1) are connected by an edge and only if $x + x_1 = yy_1$. (Technically this is a multigraph as it has loops.) Let G be the graph on $n = p^2$ vertices that we get by deleting the loops from G_0.

   (a) Prove that G_0 is p-regular and has at most p loops.
   (b) Deduce that G has $(\frac{1}{2} + o(1))n^{3/2}$ edges.
   (c) Show that any two vertices in G have at most 1 common neighbor (and hence G is $K_{2,2}$-free).