Problem Sheet 9

November 11, 2024

Question 1

Use heat.m to check the convergence results proved in the lecture for the implicit scheme when $T = 1$, $f(x, t) = 0$ and $u_0(x) = \sin(\pi x)$. (Try $heat(9, 10)$; $heat(19, 40)$; $heat(39, 160)$, ...)

Question 2

Graded Exercise for Group 2

Implement the explicit scheme for the 1D heat equation when $f(x, t) = 0$ and $u_0(x) = \sin(\pi x)$.

(a) Choose $N = 9$, $\tau = \dfrac{h^2}{2} = 0.005$ and $M = 10$. Then multiply N by 2, M by 4, divide τ by 4, and so on ...

(b) Check stability : if $\tau < \frac{h^2}{2}$, then $\lim_{n\to\infty} ||u^n||_\infty = 0$.

For instance, set $N = 19$, $\tau = \dfrac{h^2}{2} = 0.00125$, $M = 10^6$; $||u^M||$ should be close to 0. Then set $N = 19$, $\tau = \frac{h^2}{2} = 0.00126$, $M = 10^6$; $||u^M||$ should be extremely large!

Question 3

Consider the parabolic advection-diffusion problem of Sheet 8, Problem 2.
The scheme writes

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{\tau} + A\vec{u}^{n+1} = \vec{1},$$

where $A = \epsilon A_1 + A_2$,

$$A_1 = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \qquad A_2 = \frac{1}{2h}\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & -1 \\ & & 1 & 0 \end{pmatrix}.$$

We know that, for every $v \in \mathbb{R}^N$, $\lambda_1 v^T v \leqslant v^T A_1 v \leqslant \lambda_N v^T v$, where $\lambda_i = 2\dfrac{1 - \cos(i\pi h)}{h^2}$, $i = 1 \ldots, N$.

1. Prove that, for every $v \in \mathbb{R}^N$, $v^T A_2 v = 0$.

2. Using $(\vec{a} - \vec{b})^T \vec{a} = \dfrac{1}{2}\big(||\vec{a}||_2^2 - ||\vec{b}||_2^2 + ||\vec{a} - \vec{b}||_2^2\big)$, prove that

$$||\vec{u}^{n+1}||_2^2 + \tau\epsilon(\vec{u}^{n+1})^T A_1 \vec{u}^{n+1} \leqslant \frac{\tau||\vec{1}||_2^2}{\lambda_1 \epsilon} + ||\vec{u}^n||_2^2$$

so that

$$||\vec{u}^M||_2^2 + \tau\epsilon \sum_{n=0}^{M-1} (\vec{u}^{n+1})^T A_1 \vec{u}^{n+1} \leqslant T\frac{||\vec{1}||_2^2}{\lambda_1 \epsilon}.$$

Question 4

Consider the 2d parabolic convection-diffusion problem ($\epsilon > 0$)

$$\begin{cases} \dfrac{\partial u}{\partial t}(x_1, x_2, t) - \epsilon \dfrac{\partial^2 u}{\partial x^2}(x_1, x_2, t) - \dfrac{\partial u}{\partial x_1}(x_1, x_2, t) = 1, & (x_1, x_2) \in]0, 1[^2,\ 0 < t \leqslant 10; \\ u(x_1, x_2, t) = 0, & (x_1, x_2) \in \partial(]0, 1[^2), \\ 0 < t \leqslant 10; & \\ u(x_1, x_2, 0) = 0, & (x_1, x_2) \in \partial(]0, 1[^2). \end{cases} \tag{1}$$

The matlab file `diffconv2dparab.m` implements a centered finite difference method for solving the above problem. Let L be the number of internal points in the $]0, 1[$ interval, let $h = \frac{1}{L+1}$ be the space step. Let M be the number of time steps $\tau = \frac{T}{M}$ the time step.
When running the file, approximation of $u(\frac{i}{L+1}, \frac{j}{L+1}, \frac{n}{M})$ are provided, $i, j = 1, ..., L$, $n = 1, ..., M$.

Fill the matlab file.
Provide an approximation of $||u(\cdot, \cdot, 10)||_{L^\infty(]0,1[^2)}$ with three digits when $\epsilon = 0.1$.