EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2024

Problem Sheet 7

November 4, 2024

Question 1

Graded exercise for group 3

Let $\epsilon > 0$ (small), we are looking for $u : [0,1] \to \mathbb{R}$ such that

$$-\epsilon u''(x) + u'(x) = 1, \qquad 0 < x < 1, \tag{1}$$

$$u(0) = u(1) = 0. \tag{2}$$

- Check that $u(x) = x - \frac{1-e^{\frac{x}{\epsilon}}}{1-e^{\frac{1}{\epsilon}}}$ and plot the solution for $\epsilon = 0.01$. **[Not graded]**
- Let N be a positive integer, $h = \frac{1}{N+1}$, $x_i = ih$, $i = 0, 1, ..., N, N+1$. Consider the centered schema

$$-\epsilon \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{u_{i+1} - u_{i-1}}{2h} = 1, \ i = 1, \dots, N, \tag{3}$$

$$u_0 = u_{N+1} = 0. \tag{4}$$

  Write the schema as a linear system $A\vec{u} = \vec{1}$ (define A).
- Let $\vec{U}$ be the vector of exact solution $u(x_i)$, $i = 1, ..., N$. Prove that $A\vec{U} = \vec{1} + \vec{r}$ where $|r_i| \leqslant Ch^2$, $i = 1, ..., N$, with C independent of h.
- Prove that if $h \leqslant 2\epsilon$ then $A\vec{w} \geqslant \vec{1}$ where $w_i = x_i$, $i = 1, ..., N$.
- Prove that if $h \leqslant 2\epsilon$ then $A\vec{z} \geqslant \vec{0}$ implies $\vec{z} \geqslant \vec{0}$ (use Question 2 of Sheet 6).
- Let $\vec{g} \in R^N$ and $\vec{v}$ such that $A\vec{v} = \vec{g}$. Then prove that $\|\vec{v}\|_\infty \leqslant \|\vec{g}\|_\infty$.
  Prove that there exists $\tilde{C} > 0$ such that $\forall 0 < h < 2\epsilon$

$$\|\vec{U} - \vec{u}\|_\infty \leqslant Ch^2. \tag{5}$$

- Implement the problem. **[Not graded]**
  - **(a)** For $\epsilon = 0.01$ and $N = 9, 19, 39, 79, 159, 319, \dots$ check the convergence of the method.
  - **(b)** Take $N = 39$. What happens for $\epsilon = 0.1, 0.01, 0.001$?

Question 2

The file `gradient.m` implements the gradient method with optimal step for the $L \times L$ matrix

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -1 & 2 & -1 \end{pmatrix}.$$

Check that the number of iterations is $\mathcal{O}(L^2)$.

Question 3

The file `conjgrad.m` implements the Laplace problem in dimensions $d = 1, 2$ or 3 with finite differences, the linear system system being solved with the conjugate gradient method. L denotes the number of points in each direction.

Check the cpu time and memory requirement seen during the lecture and compare those with the Cholesky direct method. You can fill the following table :

	memory		cpu time		iteration_number
L	CG	Cholesky	CG	Cholesky	CG
10					
20					
40					
80					