EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2024

Problem Sheet 4

September 30, 2024

Question 1

Graded exercise for group 3

Consider an s-stages implicit Runge-Kutta method for solving $y'(t) = f(t, y(t))$, $y(t_0) = y_0 \in \mathbb{R}^d$

$$k_i = f(t_0 + c_i h, y_0 + h \sum_{j=1}^{s} a_{ij} k_j) \quad i = 1, ..., s, \tag{1}$$

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i k_i$$

Here $c_i, b_i, a_{ij} \in \mathbb{R}$ and satisfy $c_1 = 0$, $c_i = \sum_{j=1}^{s} a_{ij}$.

Assume $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous (with constant L and norm $|| \cdot ||$) with respect to the second variable. Prove that there exists a unique solution of (1) if $h < \frac{1}{L \max\limits_{1 \leqslant i \leqslant s} \sum_{j=1}^{s} |a_{ij}|}$.

Indication : consider the fixed point iteration

$$k_i^{(m+1)} = f(t_0 + c_i h, y_0 + h \sum_{j=1}^{s} a_{ij} k_j^{(m)}) \quad m = 0, 1, 2, ... \tag{2}$$

and use Banach fixed point theorem.

Question 2

Implement the adaptive algorithm that was introduced in the lecture and apply it to ordinary differential equation given by

$$\begin{cases} \dot{y}(t) = -50\,(y(t) - cos(t))\,, & 0 < t \leqslant T, \\ y(0) = 0.15, \end{cases} \tag{3}$$

with $T = 7$. Compute y_n with the Runge method and $\hat{y}_n$ with the Forward Euler scheme.

Question 3

Prove the following Lemma of the lecture. Let y be the solution to the ODE

$$\begin{cases} y'(t) = f(t, y(t)), & t_0 \leqslant t \leqslant T, \\ y(t_0) = y_0, \end{cases}$$

and y_n a RK method given by

$$\begin{aligned} &y_{n+1} = y_n + h_n \Phi(t_n, y_n, h_n), \\ &t_n = t_{n-1} + h_n, \qquad n = 0, 1, 2, ..., N, \end{aligned}$$

where $\Phi(t, z, h)$ is defined as in the lecture.

If $f(t, z)$ satisfies a Lipschitz condition, that is there exists $L > 0$ such that

$$\|f(t, z_1) - f(t, z_2)\| \leqslant L\|z_1 - z_2\|, \qquad \forall (t, z_1), (t, z_2) \text{ in a neighbourhood of } (t, y(t)),$$

then there exists $\Lambda > 0$ such that

$$\|\Phi(t, z_1, h) - \Phi(t, z_2, h)\| \leqslant \Lambda \|z_1 - z_2\|, \qquad \forall t \leqslant T, 0 < h \leqslant h_{max}.$$