Problem Sheet 2

September 20, 2024

Question 1

Given a $N \times N$ symmetric positive definite matrix A, given $\vec{X_0}$, $\vec{V_0} \in \mathbb{R}^N$ we consider the differential system :

$$\begin{cases} \vec{X}'(t) = \vec{V}(t), \\ \vec{V}'(t) = -A\vec{X}(t), \qquad 0 < t \leqslant T, \\ \vec{X}(0) = \vec{X}_0, \qquad \vec{V}(0) = \vec{V}_0. \end{cases} \tag{1}$$

1. Check that :

$$||\vec{V}(T)||^2 + \vec{X}(T)^T A\vec{X}(T) = ||\vec{V}_0||^2 + \vec{X}_0^T A\vec{X}_0$$

2. Let N be a positive integer, let $h = \frac{T}{N}$ be the time step and $t_n = nh$ where $n = 0, 1, ..., N$. Consider the Crank-Nicolson scheme for solving (1) :

$$\begin{cases} \dfrac{\vec{X}_{n+1} - \vec{X}_n}{h} = \dfrac{\vec{V}_{n+1} + \vec{V}_n}{2}, \\ \\ \dfrac{\vec{V}_{n+1} - \vec{V}_n}{h} = -A\dfrac{\vec{X}_{n+1} + \vec{X}_n}{2}, \quad n = 0, 1, ..., N-1. \end{cases} \tag{2}$$

Check that :

$$||\vec{V}_N||^2 + (\vec{X}_N)^T A\vec{X}_N = ||\vec{V}_0||^2 + \vec{X}_0^T A\vec{X}_0.$$

3. Check that (1) can be rewritten as :

$$\begin{cases} \vec{X}''(t) + A\vec{X}(t) = 0, \\ \vec{X}(0) = \vec{X}_0 \qquad \vec{X}'(0) = \vec{V}_0, \end{cases} \tag{3}$$

and that (2) can be rewritten as the Newmark scheme :

$$\frac{\vec{X}_{n+1} - 2\vec{X}_n + \vec{X}_{n-1}}{h^2} + A\frac{\vec{X}_{n+1} + 2\vec{X}_n + \vec{X}_{n-1}}{4} = 0.$$

for $n = 1, ..., N-1$. How can $\vec{X}_1$ be computed ?