Problem Sheet 8

November 4, 2024

Question 1

1. Show the Cauchy-Schwarz inequality : $\forall f, g \in \mathcal{C}([0,1])$,

$$\int_0^1 f(x)g(x)dx \leqslant \left(\int_0^1 (f(x))^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 (g(x))^2 dx\right)^{\frac{1}{2}}.$$

Indication : Consider $t \in \mathbb{R}$ and

$$\int_0^1 \Big(tf(x) + g(x)\Big)^2 dx \geqslant 0.$$

2. Show the Poincaré inequality : $\forall v \in \mathcal{C}^1([0,1])$ with $v(0) = 0$ or $v(1) = 0$,

$$\int_0^1 (v(x))^2 dx \leqslant \int_0^1 (v'(x))^2 dx.$$

Question 2

Graded Exercise for Group 1

Consider the time dependent advection-diffusion problem : Given $T, \epsilon > 0$, find $u : [0,1] \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \epsilon \dfrac{\partial^2 u}{\partial x^2}(x,t) - \dfrac{\partial u}{\partial x}(x,t) = 1, & 0 < x < 1,\ t > 0; \\ u(0,t) = 0, & t > 0; \\ u(1,t) = 0, & t > 0; \\ u(x,0) = 0, & 0 < x < 1. \end{cases} \tag{1}$$

Let M, N be large integers, $h = \dfrac{1}{N+1}$, $\tau = \dfrac{T}{M}$, $x_i = ih$, $i = 0, 1, \ldots, N+1$, $t_n = n\tau$, $n = 0, 1, \ldots, M$.

(a) Proceed as in the lecture to write a difference scheme (write the PDE at time t_{n+1}, use a backward Finite Difference Formula for $\dfrac{\partial u}{\partial t}$ and a centered FDF for $\dfrac{\partial u}{\partial x}, \dfrac{\partial^2 u}{\partial x^2}$).
Write the scheme as

$$(I + \tau A)\vec{u}^{n+1} = \tau\vec{1} + \vec{u}^n,\ \ n = 0, 1, \ldots, M-1,$$

where $\vec{u}^n$ has components u_i^n, $i = 1, \ldots, N$, which are approximations of $u(x_i, t_n)$ and where A has to be defined.

Assume $h \leqslant 2\epsilon$.

(b) Prove that $||\vec{u}^{n+1}||_\infty \leqslant ||\vec{u}^n||_\infty + \tau$, so that $||\vec{u}^M||_\infty \leqslant T$.

(c) Prove that $u_i^n \geqslant 0$, $i = 1, \ldots, N$, $n = 1, \ldots, M$.

(d) Let $\vec{U}^n \in \mathbb{R}^N$ with components $u(x_i, t_n)$. Prove that, under reasonable assumptions on u,

$$(I + \tau A)\vec{U}^{n+1} = \tau\vec{1} + \vec{U}^n + \tau\vec{r}^{n+1},$$

where $||\vec{r}^{n+1}||_\infty \leqslant C(h^2 + \tau)$, with C independent of h and τ.

(e) Prove that $||\vec{U}^M - \vec{u}^M||_\infty \leqslant CT(h^2 + \tau)$.

Question 3

Consider the 2D diffusion-convection problem :

$$-\epsilon\Delta u(x_1,x_2)+\frac{\partial u}{\partial x_1}(x_1,x_2)=1\ (x_1,x_2)\in]0,1[^2 \quad (2)$$

$$u(x_1,x_2)=0\ (x_1,x_2)\in \partial]0,1[^2 \quad (3)$$

implemented with order two centerred finite differences schema.

- Fill the `diffconv2d.m` file to build the matrix of the linear system using only L (the number of points in each dimension), I (the identity $L\times L$ matrix) and E (the subdiagonal $L\times L$ matrix).
- Check the results when $\epsilon=0.01$ and $L=10,20,40,80,160$. Check that the number of iterations of the GMRES solver is $O(L)$.