EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2024

Problem Sheet 6

October 14, 2024

Question 1

Let $f \in \mathcal{C}^0([0,1])$ and let $u \in \mathcal{C}^4([0,1])$ be such that

$$-u''(x) = f(x), \qquad 0 < x < 1, \tag{1}$$

$$u(0) = 0,\ u'(1) = 0. \tag{2}$$

Let $N > 0$, $h = \frac{1}{N+1}$ and $x_i = ih$, $i = 0, 1, ..., N+1$. Let u_i be the approximated value of $u(x_i)$, $i = 1, \dots, N+1$. The finite difference method leads to the linear system $A\vec{u} = \vec{f}$, where $\vec{f} \in \mathbb{R}^{N+1}$, $A \in \mathbb{R}^{(N+1)\times(N+1)}$ are defined by

$$A = \begin{pmatrix} \frac{2}{h^2} & -\frac{1}{h^2} & & & \\ -\frac{1}{h^2} & \frac{2}{h^2} & & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} \\ & & & -\frac{1}{h} & \frac{1}{h} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \\ u_{N+1} \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \\ \frac{h}{2} f(x_{N+1}) \end{pmatrix}.$$

Extend the convergence proof of the lecture to this particular case.
Hint : Use the fact that $\vec{w} \in \mathbb{R}^N$ defined by $w_i = \frac{x_i(4-x_i)}{2}$ satisfies $A\vec{w} \geqslant 1$ and Question 2, and prove that $A\vec{U} = \vec{f} + \vec{r}$, where $\vec{U} \in \mathbb{R}^{N+1}$ has components $u(x_i)$ and

$$\vec{r} = \begin{pmatrix} -\frac{h^2}{24}\left(u^{(4)}(\alpha_1) + u^{(4)}(\beta_1)\right) \\ -\frac{h^2}{24}\left(u^{(4)}(\alpha_2) + u^{(4)}(\beta_2)\right) \\ \vdots \\ -\frac{h^2}{24}\left(u^{(4)}(\alpha_N) + u^{(4)}(\beta_N)\right) \\ \frac{h^2}{6} u'''(\alpha_{N+1}) \end{pmatrix}.$$

Question 2

Let

$$A = \begin{pmatrix} \alpha_1 & -\gamma_1 & & & \\ -\beta_1 & \alpha_2 & -\gamma_2 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -\beta_{N-2} & \alpha_{N-1} & -\gamma_{N-1} \\ & & & -\beta_{N-1} & \alpha_N \end{pmatrix}.$$

with $\alpha_i > 0$, $i = 1, ..., N$ and $\beta_i > 0$, $\gamma_i \geqslant 0$, $i = 1, ..., N-1$ and $\alpha_i \geqslant \beta_{i-1} + \gamma_i$, $i = 2, ..., N-1$.
Also $\alpha_1 > \gamma_1$ and $\alpha_N \geqslant \beta_{N-1}$.
Prove that $Az \geqslant 0 \Rightarrow z \geqslant 0$.

Question 3

Graded exercise for group 2

Consider the problem, given $f : [0,1] \to \mathbb{R}$, find $u : [0,1] \to \mathbb{R}$

$$\begin{cases} -u''(x) = f(x) \quad 0 < x < 1, \\ u(0) = 0 \quad u(1) = 0. \end{cases} \tag{3}$$

Let N be a postive integer, $h = \frac{1}{N+1}$, $x_i = ih$, $i = 0, 1, ..., N+1$, and the schema (Numerov 1924)

$$\begin{cases} \frac{-u_{i-1}+2u_i-u_{i+1}}{h^2} = \frac{f(x_{i-1})+10f(x_i)+f(x_{i+1})}{12} \quad i = 1, ..., N, \\ u_0 = 0 \quad u_{N+1} = 0. \end{cases} \tag{4}$$

Prove that $\forall f \in C^4[0,1]$, $\exists C > 0$, $\forall 0 < h \leqslant 1$,

$$\max_{1 \leqslant i \leqslant N} |u(x_i) - u_i| \leqslant Ch^4. \tag{5}$$