EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2024

Problem Sheet 3

September 27, 2024

Question 1

Consider the Kepler differential system you have implemented in Problem Sheet 1 :

$$\begin{aligned} \vec{x}''(t) &= \frac{-\vec{x}(t)}{\|\vec{x}(t)\|^3}, \quad 0 < t \leqslant T, \\ \vec{x}'(0) &= \vec{v}_0, \\ \vec{x}(0) &= \vec{x}_0, \end{aligned} \tag{1}$$

Given the same condition as in Problem Sheet 1, implement the RK4 method and compare the results to those obtained with the Euler scheme. You should provide a table like the one given in Answer Sheet 1.

Question 2

Let $y_0 \in \mathbb{R}^n$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be given, and let $y : \mathbb{R} \to \mathbb{R}$ be the solution of

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = y_0. \end{cases} \tag{2}$$

Let $Y(t) = \begin{pmatrix} t \\ y(t) \end{pmatrix} \in \mathbb{R}^{n+1}$ such that

$$\begin{cases} Y'(t) = F(Y(t)), \\ Y(0) = \begin{pmatrix} t_0 \\ y_0 \end{pmatrix}, \end{cases} \tag{3}$$

where $F(Y(t)) = \begin{pmatrix} 1 \\ f(t, y(t)) \end{pmatrix}$.
Apply a general s stages explicit RK method to (3) and check that it corresponds to an s stages explicit RK method for (2) provided

$$c_i = \sum_{j=1}^{i-1} a_{ij}, i = 1, ..., s \text{ and } \sum_{i=1}^{s} b_i = 1.$$

Question 3

Graded exercise for group 2

Consider the ordinary differential equation given by

$$\begin{cases} \dot{y}(t) = \lambda y(t), \qquad 0 < t \leqslant T, \\ y(0) = y_0, \end{cases} \tag{4}$$

with $\lambda < 0$. Let N be a positive integer, let $h = \frac{T}{N}$ be the time step and $t_n = nh$ where $n = 0, 1, ..., N$.
Consider now an order 4 RK scheme with 4 stages to approximate (4). Let

$$p_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

— Prove that $y_N = (p_4(\lambda h))^N y_0$.
— Prove that there exists $C > 0$ such that $\forall \lambda < 0, \forall h > 0$ such that $|p_4(\lambda h)| \leqslant 1$, $\forall T > 0$ and $\forall y_0 \in \mathbb{R}$,

$$|y(t_N) - y_N| \leqslant C|\lambda|^5 T h^4 |y_0|.$$