Problem Sheet 13

December 09, 2024

Question 1

Let $x, b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ symmetric definite positive, $B \in \mathbb{R}^{p\times n}, c \in \mathbb{R}^p$, $Ker(B^T) = 0$.
We consider the following linear system :

$$\begin{aligned} Ax^* + B^T\mu^* &= b \\ Bx^* &= c \end{aligned}$$

(a) We solve the problem with the following algorithm (**Uzawa**) :

Given μ^0, $\rho > 0$ ▷ Initialization
For k=0,1,2,...
  Find x^k such that $Ax^k + B^T\mu^k = b$ ▷ Compute solution
  $\mu^{k+1} = \mu^k + \rho(Bx^k - c)$ ▷ Update Lagrange multiplier
End

- Prove that

$$\mu^* - \mu^{k+1} = (I - \rho BA^{-1}B^T)(\mu^* - \mu^k).$$

- Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant ... \leqslant \lambda_p$ be the eigenvalues of $BA^{-1}B^T$. Prove that

$$\|\mu^* - \mu^k\| \leqslant \max_{j=1,..,p} |1 - \rho\lambda_j|^k \|\mu^* - \mu^0\|$$

and that the method converges if $\rho \leqslant 2/\lambda_p$.

(b) Prove that $\forall \mu \in \mathbb{R}^p$,

$$\mu^T BA^{-1}B^T\mu = \max_{x\in\mathbb{R}^n x\neq 0} \frac{(x^TB^T\mu)^2}{x^TAx}.$$

(c) Assume that there exist $C_1, C_2 > 0$ such that $\forall p, \forall \mu \in \mathbb{R}^p$, we have

$$C_1\|\mu\| \leqslant \max_{x\in\mathbb{R}^n x\neq 0} \frac{x^TB^T\mu}{(x^TAx)^{1/2}} \leqslant C_2\|\mu\|. \tag{1}$$

Prove that the method converges if $\rho < 2/C_2^2$.
Remark : For those who know the Stokes problem, the upper bound in (1) is easy to prove with $C_2 = 1$ since

$$\forall q \in L_0^2(\Omega), \forall v \in H_0^1(\Omega)^2, \quad \int_\Omega q\,\mathrm{div}(v) \leqslant \|q\|_{L^2}\|\nabla v\|_{L^2}.$$

The lower bound is proved as soon as there exists C_1 such that $\forall h > 0, \forall q_h \in Q_h$, there exists $v_h \in V_h$ such that

$$C_1\|q_h\|_{L^2} \leqslant \max_{v_h\in V_h v_h\neq 0} \frac{\int_\Omega q_h\,\mathrm{div}(v_h)}{\|\nabla v\|_{L^2}},$$

where V_h, Q_h are suitable Finite Element subspaces for the discrete velocity v_h and the discrete pressure p_h.

Question 2

Consider the time dependent, incompressible Navier-Stokes equations. Given $\rho, \mu, T > 0$, $\Omega \subset \mathbb{R}^3$, $\vec{w} : \Omega \to \mathbb{R}^3$. We are looking for $\vec{u} : \Omega \times (0,T) \to \mathbb{R}^3$ and $p : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\rho\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} u_i\right) - \mu \Delta u_i + \frac{\partial p}{\partial x_i} = 0, \qquad i = 1,2,3, \qquad \text{in } \Omega \times (0,T), \tag{2}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0, \qquad \text{in } \Omega \times (0,T), \tag{3}$$

$$u_i = 0, \qquad i = 1,2,3, \qquad \text{on } \partial\Omega \times (0,T), \tag{4}$$

$$u_i(\vec{x}, 0) = \vec{w}_i(\vec{x}), \qquad \forall \vec{x} \in \Omega. \tag{5}$$

Here $\vec{u}$ has components u_1, u_2, u_3.
Check that

$$\int_\Omega \frac{1}{2}\rho \left\|\vec{u}(\vec{x},T)\right\|^2 \ \mathrm{d}\vec{x} + \sum_{i=1}^{3} \int_0^T \int_\Omega \mu \left\|\nabla u_i(\vec{x},t)\right\|^2 \ \mathrm{d}\vec{x}\ \mathrm{dt} = \int_\Omega \frac{1}{2}\rho \left\|\vec{w}(\vec{x})\right\|^2 \ \mathrm{d}\vec{x}. \tag{6}$$