Final Exam		
EPFL	**Cours:**	Advanced Numerical Analysis
	Enseignant:	Prof. Marco Picasso
	Date:	24/01/2024 **Start** 9h15 - **End** 11h45

Sciper: 12345 **Etudiant:** Michele Barucca **Number:** 1

Exam instructions

- Lay your camipro card on the table.
- Do not unfasten the sheets.
- No documents, draft papers or any electronic equipments are allowed.
- Mobile phones must be switched-off and kept inside your bag.
- Draft papers if needed are at the end of each exercises and won't be corrected.
- There are 5 exercises in this exam.

Exercise 1

Let $f \in C^3(\mathbb{R})$, $y_0 \in \mathbb{R}$. Consider the differential equation :

$$\begin{cases} y'(t) = f(y(t)), \quad t > t_0, \\ y(t_0) = y_0. \end{cases} \tag{1}$$

Let $h > 0$, $t_n = t_0 + nh$, $n \geq 0$.

a. Write a s-stage Runge-Kutta method (implicit or explicit), with coefficients a_{ij}, b_i, to compute y_1, an approximation of $y(t_1)$.

b. Derive the order 2 conditions, that is to say the conditions for a_{ij} and b_i under which

$$y(t_1) - y_1 = O(h^3).$$

c. Consider an explicit Runge-Kutta method with 2 stages satisfying the order 2 conditions in the case when $f(y) = \lambda y$, $\lambda < 0$. Assume $h < -\frac{2}{\lambda}$. Prove that there exists C independent of h and n such that

$$|y(t_{n+1}) - y_{n+1}| \leq Ch^2, \ n \geq 0.$$

Answers :

Draft papers

Exercise 2

Let $u : [0,1] \to \mathbb{R}$ be such that :

$$\begin{cases} -u''(x) + xu(x) = 1, & 0 < x < 1, \\ u(0) = 0, \\ u(1) = 0. \end{cases} \tag{1}$$

Let N be a positive integer, $h = \frac{1}{N+1}$, $x_i = ih$, $i = 0, 1, ..., N+1$.

a. Write a centered finite difference schema to compute u_i approximations of $u(x_i)$, $i = 1, ..., N$. Write the scheme as

$$(A + D)\vec{u} = \vec{1}$$

where

$$A = (N+1)^2 \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & (0) \\ (0) & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

and D is a diagonal matrix.

b. Assume that the eigenvalues of A are $\lambda_i = 2\left(1 - \cos\left(\frac{i\pi}{N+1}\right)\right)(N+1)^2$, $i = 1, ..., N$. Prove that $\exists C > 0, \forall N \geq 2, \lambda_i \geq C$ and prove that $||\vec{u}||_2 \leq \frac{\sqrt{N}}{C}$ where $||\vec{u}||_2$ is the Euclidian norm in $\mathbb{R}^N$.

c. Let $\vec{U} \in \mathbb{R}^N$ with components $u(x_i)$, assume $u \in \mathcal{C}^4[0,1]$ and prove that

$$(A + D)\vec{U} = \vec{1} + \vec{r}$$

with $|r_i| \leq \frac{E}{(N+1)^2}$, $i = 1, ..., N$, with E independent of N.

d. Prove that $||\vec{U} - \vec{u}||_2 \leq \frac{E}{C(N+1)^{\frac{3}{2}}}$.

Answers :

Draft papers

Exercise 3

Consider the matlab/octave file exam1.m :

```
function exam1(L)
I = speye(L,L);
E = sparse(2:L,1:L-1,1,L,L);
T = 2*I-E-E';
A = (kron(T,I)+kron(I,T))*(L+1)^2;
b=ones([L*L,1]);
cpu1 = cputime;
R=chol(A);
x = R\(R'\b);
cpu2 = cputime;
fprintf(" cpu time to solve the linear system %e  \n",cpu2-cpu1);
whos
end
```

that provides the following results :

```
octave:1> exam1(100)
 cpu time to solve the linear system 1.925395e+00
Variables visible from the current scope:

variables in scope: exam1: /home/picasso/teaching/advna/23/exam/exam1.m

   Attr Name        Size                     Bytes  Class
   ==== ====        ====                     =====  =====
        A       10000x10000                1033608  double
        E         100x100                     2392  double
        I         100x100                     2408  double
        R       10000x10000               16081592  double
        T         100x100                     5576  double
        b       10000x1                      80000  double
        x       10000x1                      80000  double

Total is 200050003 elements using 17285600 bytes

octave:2> exam1(200)
 cpu time to solve the linear system 4.672625e+00
Variables visible from the current scope:

variables in scope: exam1: /home/picasso/teaching/advna/23/exam/exam1.m

   Attr Name        Size                     Bytes  Class
   ==== ====        ====                     =====  =====
        A       40000x40000                4147208  double
        E         200x200                     4792  double
        I         200x200                     4808  double
        R       40000x40000              128323192  double
        T         200x200                    11176  double
        b       40000x1                     320000  double
        x       40000x1                     320000  double
```

```
Total is 3200200003 elements using 133131200 bytes

octave:3> exam1(400)
 cpu time to solve the linear system 1.282546e+02
Variables visible from the current scope:

variables in scope: exam1: /home/picasso/teaching/advna/23/exam/exam1.m

   Attr Name        Size                     Bytes  Class
   ==== ====        ====                     =====  =====
        A      160000x160000              16614408  double
        E         400x400                     9592  double
        I         400x400                     9608  double
       R      160000x160000             1025286392  double
        T         400x400                    22376  double
        b      160000x1                    1280000  double
        x      160000x1                    1280000  double

Total is 51200800003 elements using 1044502400 bytes

octave:4> exam1(800)
 cpu time to solve the linear system 1.090452e+03
Variables visible from the current scope:

variables in scope: exam1: /home/picasso/teaching/advna/23/exam/exam1.m

   Attr Name        Size                     Bytes  Class
   ==== ====        ====                     =====  =====
        A      640000x640000              66508808  double
        E         800x800                    19192  double
        I         800x800                    19208  double
        R      640000x640000            8197132792  double
        T         800x800                    44776  double
        b      640000x1                    5120000  double
        x      640000x1                    5120000  double

Total is 819203200003 elements using 8273964800 bytes
```

a. Which pde is involved ? (Specify the pde, the computational domain and the boundary conditions).
b. Which finite difference scheme is used ? (Specify the grid, the scheme and the linear system).
c. What is the required memory needed to perform the Cholesky decomposition (O(L^α) for some integer α) ? Compare with experiments.
d. What is the CPU time needed to perform the Cholesky decomposition and solve the linear system on one core (O(L^β) for some integer β) ? Compare with experiments (the experiments have been performed on a 16 core machine).

Answers :

Draft papers

Exercise 4

Consider the following problem : find $u : [0,1] \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) + u(x,t) = 1, & 0 < x < 1,\ 0 < t \leq T, \\ u(0,t) = 0,\ u(1,t) = 0, & 0 < t \leq T, \\ u(x,0) = sin(\pi x), & 0 < x < 1. \end{cases}$$

Let $N,\ M$ be two positive integers, $h = \frac{1}{N+1}$, $\tau = \frac{T}{M}$, $x_i = ih$, $i = 0, ..., N+1$, $t_n = n\tau$, $n = 0, ..., M$. Let u_i^n be an approximation of $u(x_i, t_n)$ obtained using centered finite differences in space and the Euler implicit scheme in time.

a. Write the scheme (don't forget to specify the indices, boundary and initial conditions).

b. Let $\vec{u}^n \in \mathbb{R}^N$ be the vector with components u_i^n, $i = 1, ..., N$. Write the scheme as :

$$A\vec{u}^{n+1} = \vec{u}^n + \tau\vec{1},$$

where $\vec{1}$ has components 1 and A has to be specified.

c. Assume that for any symmetric matrix $B \in \mathbb{R}^{N\times N}$ we have $\forall \vec{x} \in \mathbb{R}^N$

$$||B\vec{x}||_2 \leq \max_{1\leq i\leq N} |\mu_i| ||\vec{x}||_2,$$

where μ_i are the (real) eigenvalues of B and prove that $\forall h > 0$ and $\forall \tau > 0$

$$||\vec{u}^{n+1}||_2 \leq ||\vec{u}^n||_2 + \tau||\vec{1}||_2.$$

d. Let $\vec{U}^n \in \mathbb{R}^N$ with components $u(x_i, t_n)$, $i = 1, ..., N$. Assume there exists $C > 0$ independent of h and τ such that

$$A\vec{U}^{n+1} = \vec{U}^n + \tau\vec{1} + \tau\vec{r}^{n+1},$$

with $|r_i^{n+1}| \leq C(h^2 + \tau)$, $i = 1, ..., N$, $n = 0, ..., M-1$, and prove that

$$||\vec{U}^M - \vec{u}^M||_2 \leq CT\frac{h^2 + \tau}{\sqrt{h}}.$$

Answers :

Draft papers

Exercise 5

Let $u : (0,1) \times (0,T) \to \mathbb{R}$ such that :

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) + \dfrac{\partial u}{\partial t}(x,t) = 0, & 0 < x < 1,\ 0 < t \leq T, \\ u(0,t) = 0,\ u(1,t) = 0, & 0 < t \leq T, \\ u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = v_0(x), & 0 < x < 1. \end{cases} \tag{1}$$

where $u_0, v_0 : (0,1) \to \mathbb{R}$ are given smooth functions.

a. Multiply (**??**) by $\frac{\partial u}{\partial t}(x,t)$ and prove that

$$\int_0^1 \left(\left(\frac{\partial u}{\partial t}(x,T) \right)^2 + \left(\frac{\partial u}{\partial x}(x,T) \right)^2 \right) dx \leq \int_0^1 ((v_0(x))^2 + (u_0'(x))^2) dx.$$

Let N, M be two positive integers, $h = \frac{1}{N+1}$, $x_i = ih$, $i = 0, 1, ..., N, N+1$, $\tau = \frac{T}{M}$, $t_n = n\tau$, $n = 0, ..., M$.
Let $\vec{u}^n \in \mathbb{R}^n$ be the vector of components u_i^n approximations of $u(x_i, t_n)$, $i = 1, ..., N$.
Given $\vec{u}^1$ and $\vec{u}^0$, the $\vec{u}^n$ are computed using the following scheme :

$$\frac{\vec{u}^{n+1} - 2\vec{u}^n + \vec{u}^{n-1}}{\tau^2} + A\frac{\vec{u}^{n-1} + 2\vec{u}^n + \vec{u}^{n+1}}{4} + \frac{\vec{u}^{n+1} - \vec{u}^{n-1}}{2\tau} = 0\ ,\ n = 1, ..., M-1. \tag{2}$$

b. What is A in (**??**) ?

c. Take the scalar product of (**??**) with $\vec{u}^{n+1} - \vec{u}^{n-1}$ and prove that

$$\left\| \frac{\vec{u}^{n+1} - \vec{u}^n}{\tau} \right\|^2 + \left(\frac{\vec{u}^{n+1} + \vec{u}^n}{2} \right)^T A \left(\frac{\vec{u}^{n+1} + \vec{u}^n}{2} \right) \leq \left\| \frac{\vec{u}^n - \vec{u}^{n-1}}{\tau} \right\|^2 + \left(\frac{\vec{u}^n + \vec{u}^{n-1}}{2} \right)^T A \left(\frac{\vec{u}^n + \vec{u}^{n-1}}{2} \right).$$

Answers :

Draft papers