EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2024

Answer Key 9

November 11, 2024

Question 1

The L^∞ error converges as $\mathcal{O}(h^2+\tau)$ (the error is divided by $2^2=4$ each time) whereas the L^2 error converges as $\mathcal{O}\Big(\frac{h^2+\tau}{h^{\frac{1}{2}}}\Big)$ (the error is divided by $2^{\frac{3}{2}}$ each time).

N	M	err$_{inf}$	err$_2$
9	10	$2.03 \cdot 10^{-2}$	$4.54 \cdot 10^{-2}$
19	40	$5.24 \cdot 10^{-3}$	$1.66 \cdot 10^{-2}$
39	160	$1.32 \cdot 10^{-3}$	$5.90 \cdot 10^{-3}$
79	640	$3.31 \cdot 10^{-4}$	$2.09 \cdot 10^{-3}$

Question 2

(a) The L^∞ error converges as $\mathcal{O}(h^2+\tau)$ (the error is divided by $2^2=4$ each time) whereas the L^2 error converges as $\mathcal{O}\Big(\frac{h^2+\tau}{h^{\frac{1}{2}}}\Big)$ (the error is divided by $2^{\frac{3}{2}}$ each time).

N	M	err$_{inf}$	err$_2$
9	10	$5.07 \cdot 10^{-3}$	$1.33 \cdot 10^{-2}$
19	40	$1.25 \cdot 10^{-3}$	$3.94 \cdot 10^{-3}$
39	160	$3.10 \cdot 10^{-4}$	$1.39 \cdot 10^{-3}$
79	640	$7.74 \cdot 10^{-5}$	$4.90 \cdot 10^{-4}$

(b) For $N=19$, $\tau=0.00125$ and $M=10^6$, $||\vec{u}^M||_\infty$ is approximately equal to zero. For $N=19$, $\tau=0.00126$ and $M=10^6$ $u_i^M = Inf$.

Question 3

(a)

$$\begin{aligned}\vec{v}^T A_2 \vec{v} &= (v_1,\ v_2,\ \ldots,\ v_N)^T(-v_2,\ v_1-v_3,\ \ldots,\ v_{N-2}-v_N,\ v_{N-1})\\ &= -v_1 v2 + v_1 v_2 - v_3 v_2 + \ldots + v_{N-1}v_N\\ &= 0.\end{aligned}$$

Remark that, for every $B\in\mathbb{R}^{N\times N}$ satisfying $B^T=-B$, we have $\vec{v}^T B\vec{v} = 0\ \forall \vec{v}\in\mathbb{R}^N$.

(b) Using the scheme that $\vec{u}$ satisfies and the indication, we have

$$\begin{aligned}||\vec{u}^{n+1}||_2^2 - ||\vec{u}^n||_2^2 + ||\vec{u}^{n+1}-\vec{u}^n||_2^2 + 2\tau(\vec{u}^{n+1})^T A\vec{u}^{n+1} &= 2\tau(\vec{u}^{n+1})^T\vec{1}\\ &\leqslant \tau\epsilon\lambda_1||\vec{u}^{n+1}||_2^2 + \frac{\tau}{\epsilon\lambda_1}||\vec{1}||_2^2,\end{aligned}$$

where we have used Young inequality.
From point 1., $\epsilon(\vec{u}^{n+1})^T A\vec{u}^{n+1} = \epsilon(\vec{u}^{n+1})^T A_1\vec{u}^{n+1}$. Using $\lambda_1||\vec{u}^{n+1}||_2^2 \leqslant (\vec{u}^{n+1})^T A_1 \vec{u}^{n+1}$ and $||\vec{u}^{n+1}-\vec{u}^n||_2^2 \geqslant 0$, we thus obtain

$$||\vec{u}^{n+1}||_2^2 - ||\vec{u}^n||_2^2 + \tau\epsilon(\vec{u}^{n+1})^T A_1\vec{u}^{n+1} \leqslant \tau\frac{||\vec{1}||_2^2}{\lambda_1\epsilon}.$$

Suming from $n=0$ to $M-1$, we get

$$||\vec{u}^M||_2^2 - ||\vec{u}^0||_2^2 + \tau\epsilon\sum_{n=0}^{M-1}(\vec{u}^{n+1})^T A_1\vec{u}^{n+1} \leqslant M\tau\frac{||\vec{1}||_2^2}{\lambda_1\epsilon}.$$

Since $u^0=0$ and $M\tau = T$, we get the desired inequality.

Question 4

The missing lines in the matlab file are :

— line 9 : `mat=speye(L*L,L*L)+tau*A;`.

— Line 13 : `rhs=u+tau*b;`.

— Line 15 : `u=gmres(mat,rhs,1000,1.e-6,1000,lower,upper,u);`.

To analyze the convergence of the method you should know the analytical solution of the problem you are facing. Just for the scope of clarity, in this case, since we are using a Backword FD for the time derivative and a Centered FD schema for the space derivative, we should expect a first order convergence in time and second order convergence in space.
In many situation, the analytical solution is not known. To see if the method has been arrived to convergence you can observe different quantities, one of those could be the L^∞ norm of the n solution.
You have to run many simulations to see when the norm of $||u(\cdot,\cdot,10)||_{L^\infty(]0,1[^2)}$ is changing no more. In this execrcise we fix the time step $M = 10$ and we run the simulation for different values of $L = 10, 20, 40, 80$.
For this simple problem you don't see any problem on making many simulations. But think about the case in which you have a more complex problem, how many resources and time this type of study may require.

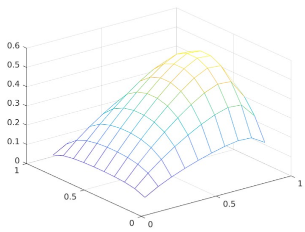

FIGURE 1 – Simulation run with $L = 10$ and $||u(\cdot,\cdot,10)||_{L^\infty(]0,1[^2)} = 0.545$.

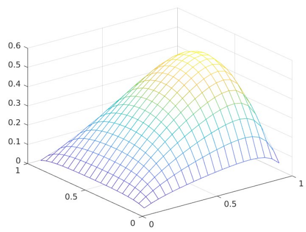

FIGURE 2 – Simulation run with $L = 20$ and $||u(\cdot,\cdot,10)||_{L^\infty(]0,1[^2)} = 0.548$.

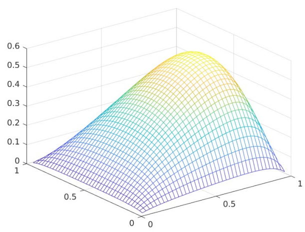

FIGURE 3 – Simulation run with $L = 40$ and $||u(\cdot,\cdot,10)||_{L^\infty(]0,1[^2)} = 0.547$.

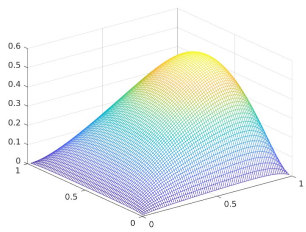

FIGURE 4 – Simulation run with $L = 80$ and $||u(\cdot,\cdot,10)||_{L^\infty(]0,1[^2)} = 0.547$.