Answer Key 8

November 4, 2024

Question 1

1. Note that the inequality is trivial if $f = 0$ or $g = 0$. Thus, we assume $f, g \neq 0$. We have

$$p(t) = t^2 \int f^2 + 2t \int fg + \int g^2 \geqslant 0 \ \forall t \in \mathbb{R}.$$

Therefore

$$0 \geqslant \Delta = 4\Big(\int fg\Big)^2 - 4 \int f^2 \int g^2.$$

2. We assume $v(0) = 0$ The proof in the case $v(1) = 0$ is similar. Note that $\forall x \in [0, 1]$,

$$v(x) = v(0) + \int_0^x v'(s)ds = \int_0^x v'(s)ds.$$

Thus

$$(v(x))^2 = \Big(\int_0^x v'(s)ds\Big)^2 \leqslant \int_0^x (v'(s))^2 ds \int_0^x 1^2 ds \leqslant \int_0^1 (v'(s))^2 ds.$$

Question 2

(a) The difference scheme reads

$$\begin{cases} \dfrac{u_i^{n+1} - u_i^n}{\tau} - \epsilon \dfrac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{h^2} - \dfrac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2h} = 1, & n = 1, \dots, M-1, \ i = 1, \dots, N; \\ u_0^n = 0, & n = 1, \dots, M-1; \\ u_{N+1}^n = 0, & n = 1, \dots, M-1; \\ u_i^0 = 0, & i = 0, \dots, N+1. \end{cases} \tag{1}$$

(b) Let k be such that $|u_k^{n+1}| \geqslant |u_i^{n+1}|$ for every $i = 1, \dots, N$. Then,

$$\Big(1 + \frac{2\tau\epsilon}{h^2}\Big)u_k^{n+1} = u_k^n + \Big(-\frac{\tau}{2h} + \frac{\tau\epsilon}{h^2}\Big)u_{k-1}^{n+1} + \Big(\frac{\tau}{2h} + \frac{\tau\epsilon}{h^2}\Big)u_{k+1}^{n+1} + \tau.$$

Since $h \leqslant 2\epsilon$, this implies

$$\Big(1 + \frac{2\tau\epsilon}{h^2}\Big)|u_k^{n+1}| \leqslant |u_k^n| + \Big(-\frac{\tau}{2h} + \frac{\tau\epsilon}{h^2}\Big)|u_k^{n+1}| + \Big(\frac{\tau}{2h} + \frac{\tau\epsilon}{h^2}\Big)|u_k^{n+1}| + \tau$$

and therefore

$$||u^{n+1}||_\infty = |u_k^{n+1}| \leqslant |u_k^n| + \tau \leqslant ||u^n||_\infty + \tau.$$

(c) Let k such that $u_k^{n+1} \leqslant u_i^{n+1}$, for every $i = 1, \dots, N$. It suffices to show that $u_k^{n+1} \geqslant 0$. Then

$$\begin{aligned} \Big(1 + \frac{2\tau\epsilon}{h^2}\Big)u_k^{n+1} &= u_k^n + \Big(-\frac{\tau}{2h} + \frac{\tau\epsilon}{h^2}\Big)u_{k-1}^{n+1} + \Big(\frac{\tau}{2h} + \frac{\tau\epsilon}{h^2}\Big)u_{k+1}^{n+1} + \tau \\ &\geqslant u_k^n + \Big(-\frac{\tau}{2h} + \frac{\tau\epsilon}{h^2}\Big)u_k^{n+1} + \Big(\frac{\tau}{2h} + \frac{\tau\epsilon}{h^2}\Big)u_k^{n+1} + \tau \end{aligned}$$

and thus

$$u_k^{n+1} \geqslant u_k^n + \tau \geqslant 0.$$

(d) Assume that $u(x,t)$ is $\mathcal{C}^4$ in space and $\mathcal{C}^2$ in time. Then

$$\frac{U_i^{n+1}-u_i^n}{\tau}-\epsilon\frac{U_{i-1}^{n+1}-2U_i^{n+1}+U_{i+1}^{n+1}}{h^2}-\frac{U_{i+1}^{n+1}-U_{i-1}^{n+1}}{2h}=1+r_i^{n+1},$$

with

$$r_i^{n+1}\leqslant\frac{h^2}{12}\max_{(x,t)\in[0,1]\times[0,T]}|\frac{\partial^4 u}{\partial x^4}(x,t)|+\frac{\tau}{2}\max_{(x,t)\in[0,1]\times[0,T]}|\frac{\partial^2 u}{\partial t^2}(x,t)|+\frac{h^2}{6}\max_{(x,t)\in[0,1]\times[0,T]}|\frac{\partial^3 u}{\partial x^3}(x,t)|.$$

(e) Setting $\vec{e}^n=\vec{U}^n-\vec{u}^n$, we have

$$(I+\tau A)\vec{e}^{n+1}=\vec{e}^n+\tau\vec{r}^{n+1}$$

and thus

$$||\vec{e}^{n+1}||_\infty\leqslant||\vec{e}^n||_\infty+\tau||\vec{r}^{n+1}||_\infty\leqslant||\vec{e}^n||_\infty+\tau C(h^2+\tau),$$

which implies

$$||\vec{e}^M||_\infty\leqslant||\vec{e}^0||_\infty+C\underbrace{\tau M}_{T}(h^2+\tau).$$

Question 3

— Concerning the matlab file the A matrix is defined as :
`A = (kron(T,I)+kron(I,T))*eps*(L+1)^2+kron(I,E'-E)*(L+1)/2.`

— The results for $\epsilon=0.01$:

L	iteration_number
10	46
20	47
40	69
80	129
160	255

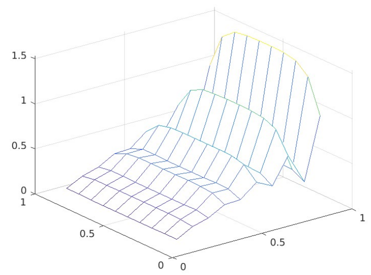

FIGURE 1 – Reasult for $\epsilon=0.01$ and $L=10$.

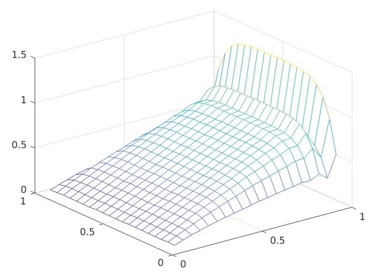

FIGURE 2 – Reasult for $\epsilon=0.01$ and $L=20$.

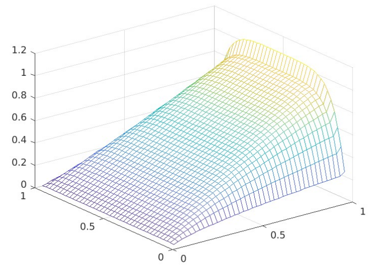

FIGURE 3 – Reasult for $\epsilon = 0.01$ and $L = 40$.

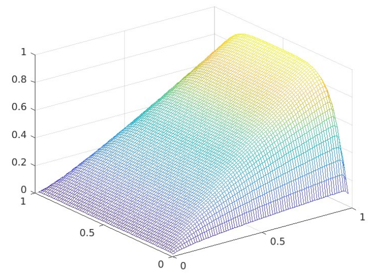

FIGURE 4 – Reasult for $\epsilon = 0.01$ and $L = 80$.

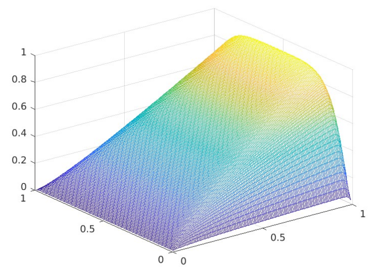

FIGURE 5 – Reasult for $\epsilon = 0.01$ and $L = 160$.

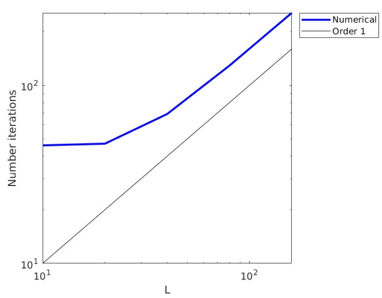

FIGURE 6 – Order of the number of iterations.