Answer Key 7

November 4, 2024

Question 1

— A is given by :

$$A = \begin{pmatrix} \frac{2\epsilon}{h^2} & -\frac{\epsilon}{h^2}+\frac{1}{2h} & & & 0 \\ -\frac{\epsilon}{h^2}-\frac{1}{2h} & \frac{2\epsilon}{h^2} & & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -\frac{\epsilon}{h^2}-\frac{1}{2h} & \frac{2\epsilon}{h^2} & -\frac{\epsilon}{h^2}+\frac{1}{2h} \\ & & & -\frac{\epsilon}{h^2}-\frac{1}{2h} & \frac{2\epsilon}{h^2} \end{pmatrix}.$$

— From 3 we can analyze separatly the two terms : $-\epsilon\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}$ and $\frac{u_{i+1}-u_{i-1}}{2h}$. Concerning the first one holds the results seen during the lecture. Concerning the second one we proceed by first developing the Taylor expansions of u_{i+1} and u_{i-1} :

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{6}u'''(\alpha_i)\ , x_i < \alpha_i < x_{i+i}, \tag{1}$$

$$u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{6}u'''(\beta_i)\ , x_{i-1} < \beta_i < x_i. \tag{2}$$

Then you take the difference :

$$\frac{u(x_{i+1}) - u(x_{i-1})}{2h} = u'(x_i) + \frac{h^3}{12h}(u'''(\alpha_i) + u'''(\beta_i)). \tag{3}$$

$$\tag{4}$$

Putting togheter the two terms for the residual you get that :

$$|r_i| \leqslant h^2(\frac{\epsilon}{12}\max_{0\leqslant x\leqslant 1}|u^{(4)}(x)| + \frac{1}{6}\max_{0\leqslant x\leqslant 1}|u'''(x)|). \tag{5}$$

— We have :

$$-\epsilon\frac{w_{i+1}-2w_i+w_{i-1}}{h^2} + \frac{w_{i+1}-w_{i-1}}{2h} = 1,\ i = 1,\dots,N-1, \tag{6}$$

whereas on the last $w(N+1) = 1$:

$$-\epsilon\frac{w_{N-1}-2w_N}{h^2} - \frac{w_{N-1}}{2h} = 1 + \frac{\epsilon}{h^2} - \frac{1}{2h} \geqslant 1 \tag{7}$$

if $\frac{\epsilon}{h^2} - \frac{1}{2h} \geqslant 0$.

— Check that you can use the Lemma of Question 2 Sheet 6 :

$$A = \begin{pmatrix} \alpha_1 & -\gamma_1 & & & 0 \\ -\beta_1 & \alpha_2 & -\gamma_2 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -\beta_{N-2} & \alpha_{N-1} & -\gamma_{N-1} \\ & & & -\beta_{N-1} & \alpha_N \end{pmatrix}.$$

Since $\alpha_i = \frac{2\epsilon}{h^2} > 0$, $\beta_i = \frac{\epsilon}{h^2} + \frac{1}{2h} > 0$ and $\gamma_i = \frac{\epsilon}{h^2} - \frac{1}{2h} \geqslant 0$. Then $A\vec{z} \geqslant \vec{0}$ implies $\vec{z} \geqslant \vec{0}$.

— Since the maximum of $w_i = x_i$ is equal to 1. Then $\|\vec{g}\|_\infty = \max_{1\leqslant i\leqslant N} \vec{g} = 1$.
We have first to see if the Lemma saw during the lecture hold :

$$A\vec{w}\|\vec{g}\|_\infty = \vec{1}\|\vec{g}\|_\infty \tag{8}$$

also you have $A\vec{v} = \vec{g}$. Then :

$$A(\vec{w}\|\vec{g}\|_\infty - \vec{v}) = \vec{1}\|\vec{g}\|_\infty - \vec{g} \geqslant \vec{0}, A(\vec{w}\|\vec{g}\|_\infty + \vec{v}) = \vec{1}\|\vec{g}\|_\infty + \vec{g} \geqslant \vec{0}. \tag{9}$$

Thus : $-\vec{w}\|\vec{g}\|_\infty \leqslant \vec{v} \leqslant \vec{w}\|\vec{g}\|_\infty$.
That is : $|\vec{v}| \leqslant |\vec{w}|\|\vec{g}\|_\infty \leqslant \|\vec{g}\|_\infty$
Then taking the quantity $A(\vec{U} - \vec{u}) = \vec{r}$, by applying the Lemma you get :

$$\|\vec{U} - \vec{u}\|_\infty \leqslant \|\vec{r}\|_\infty \leqslant Ch^2, \tag{10}$$

where $C = \frac{\epsilon}{12} \max_{0\leqslant x\leqslant 1} |u^{(4)}(x)| + \frac{1}{6} \max_{0\leqslant x\leqslant 1} |u'''(x)|$.

— **(a)** We obtain the following results : .

N	9	19	39	79	159	319
$\left\|\vec{U} - \vec{u}\right\|_\infty$	$6.96 \cdot 10^{-1}$	$4.35 \cdot 10^{-1}$	$1.93 \cdot 10^{-1}$	$5.57 \cdot 10^{-2}$	$1.21 \cdot 10^{-2}$	$3.02 \cdot 10^{-3}$

As expected, $\left\|\vec{U} - \vec{u}\right\|_\infty = \mathcal{O}(h^2)$ when h is sufficiently small.

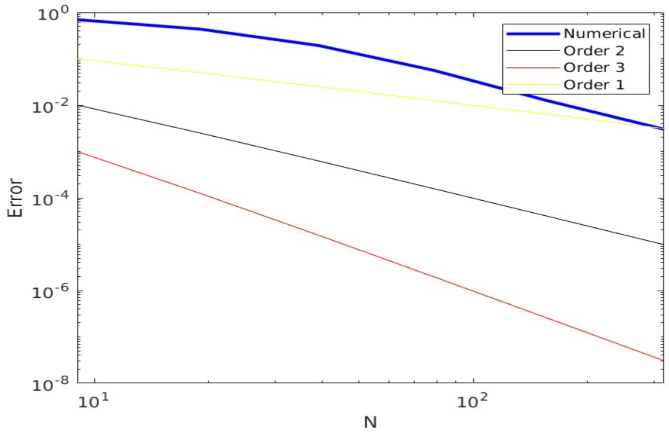

FIGURE 1 – Order of convergence.

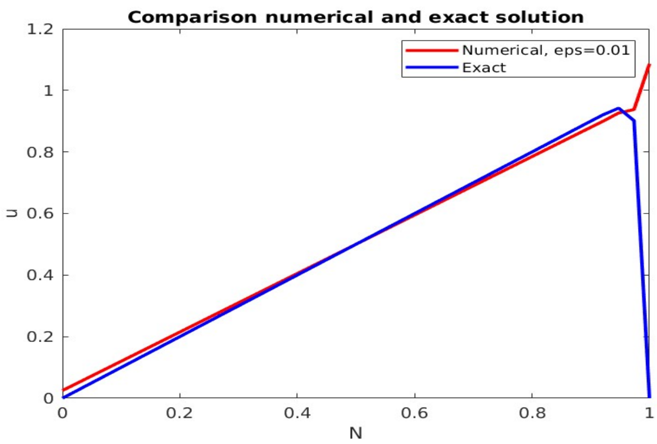

FIGURE 2 – Comparison of the real and numerical solution $\epsilon = 0.01$.

(b) For $N = 39$, the solution oscillates when $\epsilon = 0.001$.

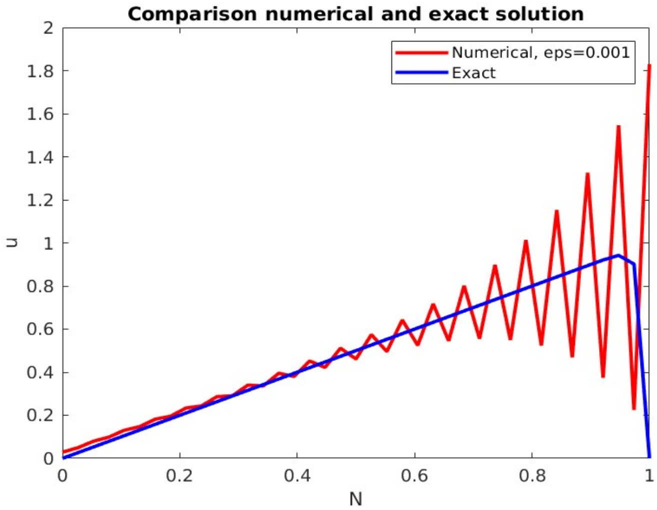

FIGURE 3 – Comparison of the real and numerical solution $\epsilon = 0.001$.

Question 2

Running the code **gradient.m** for different values of L yields the table above, which corresponds to a $\mathcal{O}(L^2)$ growth.

L	10	20	40	80	160
iteration number	339	1349	5357	21505	86993

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Question 3

We obtain the following tables :

d = 1

	memory		cpu time		iteration_number
L	CG	Cholesky	CG	Cholesky	CG
10	536	392	0.000000e+00	5.000000e-02	5
20	1096	792	0.000000e+00	3.000000e-02	10
40	2216	1592	1.000000e-02	7.000000e-02	20
80	4456	3192	1.000000e-02	3.000000e-02	40
160	8936	6392	1.000000e-02	2.000000e-02	80

d = 2

	memory		cpu time		iteration_number
L	CG	Cholesky	CG	Cholesky	CG
10	9768	16952	1.000000e-02	2.000000e-02	14
20	40328	131512	1.000000e-02	6.000000e-02	32
40	163848	1037432	0.000000e+00	2.000000e-02	63
80	660488	8244472	7.000000e-02	2.000000e-02	127
160	2652168	65743352	1.890000e+00	1.400000e+00	255

d = 3

	memory		cpu time		iteration_number
L	CG	Cholesky	CG	Cholesky	CG
10	126408	1478552	1.000000e-02	8.000000e-02	20
20	1049608	48953912	5.000000e-02	1.040000e+00	41
40	8550408	1599975032	1.630000e+00	3.058000e+01	80
80	69017608	51793818872	7.880000e+00	1.105130e+03	162
160	554598408	-	7.459000e+01	-	327

The results we obtain indeed shows that the CPU times grows with relative order $\mathcal{O}\left(L^{d+1}\right) = \mathcal{O}\left(N^{\frac{d+1}{d}}\right)$. The CPU times are too small for $d = 1, 2$ to check the complexity. When $d = 3$ the cputime from $L = 80$ to $L = 160$ is multiply by $2^4 = 16$, which corresponds to $\mathcal{O}(L^4)$.

From the lecture we know that $K(A) = \dfrac{\lambda_N}{\lambda_1} = \mathcal{O}(L^2)$, thus the number of iter of CG is $\mathcal{O}(L)$ and thus the number of operations should be $\mathcal{O}(L^{d+1}) = \mathcal{O}(N^{\frac{d+1}{d}})$.