Answer Key 6

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Question 1

We have

$$u(x_N) = u(x_{N+1}) - hu'(x_{N+1}) + \frac{h^2}{2}u''(x_{N+1}) - \frac{h^3}{6}u'''(\alpha_{N+1}),$$

for some $x_N < \alpha_{N+1} < x_{N+1}$, and thus

$$\frac{u(x_{N+1}) - u(x_N)}{h} = \frac{h}{2}f(x_{N+1}) + \frac{h^2}{6}u'''(\alpha_{N+1}).$$

Therefore, $A\vec{U} = \vec{f} + \vec{r}$, with

$$\vec{r} = \begin{pmatrix} -\frac{h^2}{24}\left(u^{(4)}(\alpha_1) + u^{(4)}(\beta_1)\right) \\ \vdots \\ -\frac{h^2}{24}\left(u^{(4)}(\alpha_N) + u^{(4)}(\beta_N)\right) \\ \frac{h^2}{6}u'''(\alpha_{N+1}) \end{pmatrix},$$

where $x_i < \alpha_i < x_{i+1}$, and $x_{i-1} < \beta_i < x_i$, $i = 1 \dots, N$.
Hence, $||\vec{r}||_\infty \leqslant Ch^2$, with $C = \max\left(\max_{0\leqslant x\leqslant 1} \frac{1}{12}\left|u^{(4)}(x)\right|, \max_{0\leqslant x\leqslant 1} \frac{1}{6}\left|u'''(x)\right|\right)$.
Then, note that A satisfies the assumption of Question 2, we thus conclude that $A\vec{v} \geqslant 0 \Rightarrow \vec{v} \geqslant 0$.

Using $A\vec{w} \geqslant \vec{1}$ and proceeding next exactly as in the lecture, we get that if $A\vec{v} = \vec{g}$, then

$$||\vec{v}||_\infty \leqslant ||\vec{w}||_\infty ||\vec{g}||_\infty \leqslant \frac{3}{2}||\vec{g}||_\infty.$$

Since $A(\vec{U} - \vec{u}) = \vec{r}$, we conclude that

$$||\vec{U} - \vec{u}||_\infty \leqslant Ch^2.$$

Question 2

Let z_k s.t. $z_k \leqslant z_i$, $i = 1, ..., N$. Then it sufficies to show that $z_k \geqslant 0$.

- If $k = 1$, $(Az)_k \geqslant 0$ writes $\alpha_1 z_1 \geqslant \gamma_1 z_1$ that is $(\alpha_1 - \gamma_1)z_1 \geqslant 0$.
  Since $\alpha_1 > \gamma_1$ this implies $z_1 \geqslant 0$.
- If $k = N$, $(Az)_k \geqslant 0$ writes $\alpha_N z_N \geqslant \beta_{N-1}z_{N-1}$.
  If $\alpha_N = \beta_{N-1}$, $z_N \geqslant z_{N-1}$ thus $z_N = z_{N-1}$.
  If $\alpha_N > \beta_{N-1}$, $\alpha_N z_N \geqslant \beta_{N-1}z_N$ that is $(\alpha_N - \beta_{N-1})z_N \geqslant 0$ so $z_N \geqslant 0$.
- If $k \in 2, ..., N-1$, $(Az)_k \geqslant 0$ writes $\alpha_k z_k \geqslant \beta_{k-1}z_{k-1} + \gamma_k z_{k+1} \geqslant \beta_{k-1}z_{k-1} + \gamma_k z_k$.
  so $(\alpha_k - \gamma_k)z_k \geqslant \beta_{k-1}z_{k-1}$.
  We know $\alpha_k - \gamma_k \geqslant \beta_{k-1}$.
  If $\alpha_k - \gamma_k = \beta_{k-1}$, $z_k \geqslant z_{k-1}$ thus $z_k = z_{k-1}$.
  if $\alpha_k - \gamma_k > \beta_{k-1}$, $(\alpha_k - \gamma_k)z_k \geqslant \beta_{k-1}z_k$ that is $(\alpha_k - \gamma_k - \beta_{k-1})z_k \geqslant 0$ so $z_k \geqslant 0$.

Repeat the process for $k-1$ instead of k and so on until we reach $k = 1$.
In all cases $z_k \geqslant 0$.

Question 3

For $i = 1, ..., N$,

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + ... + \frac{h^6}{6!}u^{(6)}(\alpha_i) \quad x_i < \alpha_i < x_{i+1}, \tag{1}$$

$$u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) + ... + \frac{h^6}{6!}u^{(6)}(\beta_i) \quad x_{i-1} < \beta_i < x_i, \tag{2}$$

then,

$$f(x_i) = -u''(x_i) = \frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} + \underbrace{2\frac{h^2}{4!}u^{(4)}(x_i)}_{=f''(x_i)} + \frac{h^4}{6!}(u^{(6)}(\alpha_i) + u^{(6)}(\beta_i)), \tag{3}$$

by rearranging the terms,

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f(x_i) + 2\frac{h^2}{12}f''(x_i) + \frac{h^4}{6!}(f^{(4)}(\alpha_i) + f^{(4)}(\beta_i)). \tag{4}$$

Notice that,

$$f''(x_i) = \frac{f(x_{i-1}) - 2f(x_i) + f(x_{i+1})}{h^2} - \frac{h^4}{4!}(f^{(4)}(\gamma_i) + f^{(4)}(\delta_i)) \qquad x_i < \gamma_i < x_{i+1} \quad x_{i-1} < \delta_i < x_i. \tag{5}$$

Thus,

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = \frac{f(x_{i-1}) + 10f(x_i) + f(x_{i+1})}{12} + r_i, \tag{6}$$

and

$$|r_i| \leqslant \frac{h^4}{6} \max_{0 \leqslant x \leqslant 1} |f^{(4)}(x)|.$$

Finaly, setting $\vec{U} = (u(x_1), ..., u(x_N))'$ you get the system of equations $A(\vec{U} - \vec{u}) = \vec{r}$ and

$$||\vec{U} - \vec{u}||_\infty \leqslant \frac{1}{8}||\vec{r}||_\infty \leqslant \underbrace{\frac{1}{48}|f^{(4)}(x)|}_{C} h^4. \tag{7}$$