Answer Key 5

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Question 1

1. Multiplying the first line by $X'(t)$

$$\begin{aligned} 0 &= mX'(t)X''(t) + kX(t)X'(t) \\ &= \frac{\mathrm{d}}{\mathrm{dt}}\Big(\frac{1}{2}m\big(X'(t)\big)^2 + \frac{1}{2}k\big(X(t)\big)^2\Big). \end{aligned}$$

2. Write the Taylor expansion for X(h) until the order two : $X(h) = X(0) + hX'(0) + \frac{h^2}{2}X''(0) + O(h^3)$. Then, by substituting the initial condition and the equation in (1), you get an expression for $X(h) = X_1$:

$$X_1 = X_0 + hV_0 - \frac{k * h^2}{2 * m}X_0 = hV_0 + (1 - \frac{k * h^2}{2 * m})X_0.$$

3. The result is obtained by multiplying the Newmark scheme by $(X_{n+1} - X_{n-1})$:

$$\begin{aligned} 0 &= \Big[m\frac{X_{n+1} - 2X_n + X_{n-1}}{h^2} + k\frac{X_{n+1} + 2X_n + X_{n-1}}{4}\Big](X_{n+1} - X_{n-1}) \\ &= \frac{1}{2}m\Big(\frac{X_{n+1} - X_n}{h}\Big)^2 - \frac{1}{2}m\Big(\frac{X_n - X_{n-1}}{h}\Big)^2 + \frac{1}{2}k\Big(\frac{X_{n+1} + X_n}{h}\Big)^2 - \frac{1}{2}k\Big(\frac{X_n + X_{n-1}}{h}\Big)^2 \end{aligned}$$

Question 2

The matrix

$$A = \frac{1}{h^2}\begin{pmatrix} a_{\frac{1}{2}} + a_{\frac{3}{2}} & -a_{\frac{3}{2}} & & & \\ -a_{\frac{3}{2}} & a_{\frac{3}{2}} + a_{\frac{5}{2}} & & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & -a_{N-\frac{1}{2}} & \\ & & & -a_{N-\frac{1}{2}} & a_{N-\frac{1}{2}} + a_{N+\frac{1}{2}} \end{pmatrix}.$$

is symmetric positive definite since, $\forall \vec{v} \in R^N$:

$$\begin{aligned} v^TAv = \frac{1}{h^2}\Big((a_{\frac{1}{2}} + a_{\frac{3}{2}})v_1^2 - 2a_{\frac{3}{2}}v_1v_2 + (a_{\frac{3}{2}} + a_{\frac{5}{2}})v_2^2 + ... - 2a_{N-\frac{1}{2}}v_{N-1}v_N + (a_{N-\frac{1}{2}} + a_{N+\frac{1}{2}})v_N^2\Big) = \\ \frac{1}{h^2}\Big(a_{\frac{1}{2}}v_1^2 + a_{\frac{3}{2}}(v_1 - v_2)^2 + a_{\frac{5}{2}}(v_2 - v_3)^2 + ... + a_{N-\frac{1}{2}}(v_{N-1} - v_N)^2 + a_{N+\frac{1}{2}}v_N^2\Big) \geqslant 0. \end{aligned}$$

Furthermore, we have $v^TAv = 0 \iff v = 0$.

Question 3

(a) Multiply (5) by $u(x)$ and integrate in the interval $0 < x < 1$

$$\int_0^1 (-u''(x)u(x) + u'(x)u(x) + u(x)u(x))dx = \int_0^1 f(x)u(x)dx$$

integrate by part the first term,

$$\int_0^1 u'(x)u'(x)dx - [u'(x)u(x)]_0^1 = \int_0^1 u'(x)u'(x)dx$$

then, the second term is euqal to zero because,

$$\int_0^1 u'(x)u(x)dx = \int_0^1 \frac{1}{2}(u^2(x))'dx = \frac{1}{2}[(u^2(x))]_0^1 = 0$$

so we get to the conclusion,

$$\int_0^1 ((u'(x))^2 + (u(x))^2)dx = \int_0^1 f(x)u(x)dx. \tag{1}$$

(b) Multiply (8) by $u(x)$ and integrate in the domain Ω. Doing similar steps to the one did before we get,

$$\int_\Omega (||\nabla u(x)||^2 + (u(x))^2)dx = \int_\Omega f(x)u(x)dx. \tag{2}$$

(c) Multiply (11) by $u(x,t)$ and integrate in the domain Ω doing similar steps to the one did for point 1 we get,

$$\int_\Omega \frac{\partial u}{\partial t}(x,t)u(x,t)dx + \int_\Omega ||\nabla u(x,t)||^2 dx = \int_\Omega \frac{1}{2}\frac{\partial u^2}{\partial t}(x,t)dx + \int_\Omega ||\nabla u(x,t)||^2 dx = 0. \tag{3}$$

by integrating in time we arrive to the final formulation,

$$\frac{1}{2}\int_\Omega (u(x,T))^2 dx + \int_0^T \int_\Omega ||\nabla u(x,t)||^2 dxdt = \frac{1}{2}\int_\Omega (u_0(x))^2 dx. \tag{4}$$