

Answer Key 4

September 30, 2024

Question 1

Proof. (2) can be written in compact form as $K^{m+1} = F(K^m)$ with $K^m = (k_1^{(m)}, \dots, k_s^{(m)})^T$ and $F_i(K) = f(t_0 + c_i h, y_0 + h \sum_{j=1}^s a_{ij} k_j) \forall K = (k_1, \dots, k_s)^T \in \mathbb{R}^{ds}$ (remember $k_i \in \mathbb{R}^d$).

Then, $\forall K = (z_1, \dots, z_s)^T, \tilde{K} = (\tilde{z}_1, \dots, \tilde{z}_s)^T \in \mathbb{R}^{ds}$,

$$\|F_i(K) - F_i(\tilde{K})\| \leq Lh \left\| \sum_{j=1}^s a_{ij}(z_j - \tilde{z}_j) \right\| \leq Lh \sum_{j=1}^s |a_{ij}| \|z_j - \tilde{z}_j\|$$

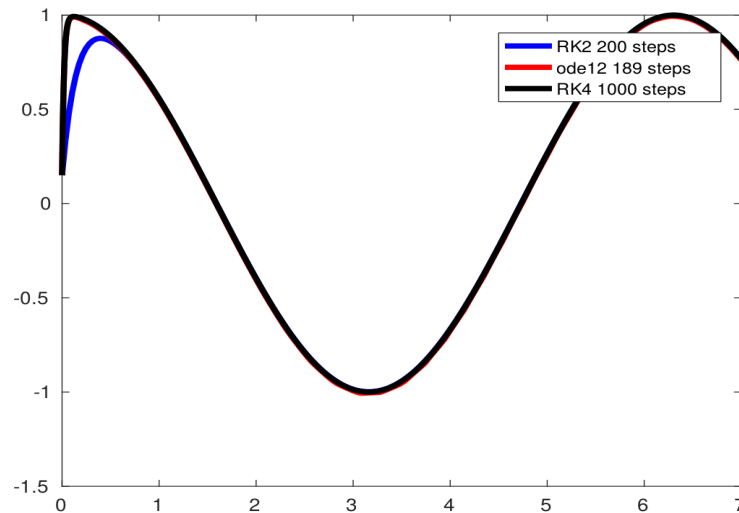
Define $\|K\| = \max_{1 \leq i \leq s} \|z_i\|$ and then summing over the indices $i = 1, \dots, s$,

$$\begin{aligned} \|F(K) - F(\tilde{K})\| &= \max_{1 \leq i \leq s} \|F_i(K) - F_i(\tilde{K})\| \leq \max_{1 \leq i \leq s} \sum_{j=1}^s Lh \sum_{j=1}^s |a_{ij}| \|z_j - \tilde{z}_j\| \\ &\leq \max_{1 \leq i \leq s} \sum_{j=1}^s Lh \sum_{j=1}^s |a_{ij}| \|K - \tilde{K}\| \end{aligned}$$

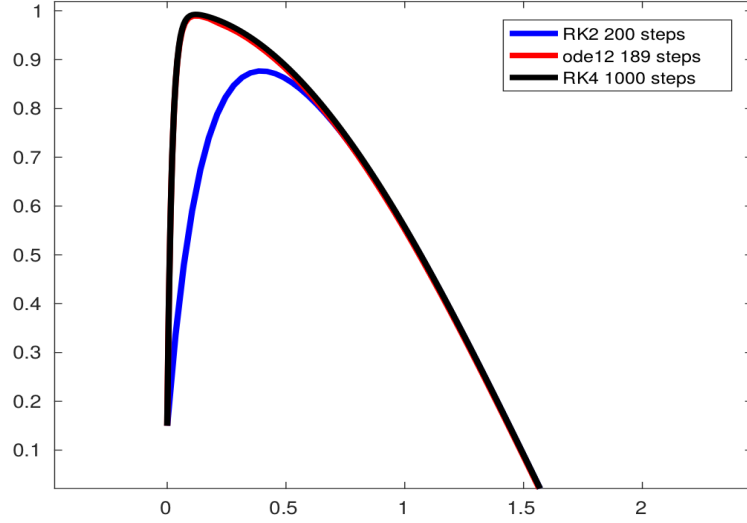
From the Banach fixed point theorem, the sequence $K^{m+1} = F(K^m)$ converge if $hL \max_{1 \leq i \leq s} \sum_{j=1}^s |a_{ij}| < 1$.

Question 2

For $T = 7$ and $y_0 = 0.15$, we have plotted the approximated solution to the given ODE using the Runge (RK2) method with a constant time step ($dt = 0.035$) and the adaptive algorithm with a tolerance = 0.01. We tend to compare the two methods by plotting the approximated solution using RK4 method with a high number of steps (1000) which is considered as very close to the exact solution.



We can observe that the adaptive algorithm with 189 steps is much more precise than Runge with 200 steps.



Question 3

We recall that $\Phi(t, z, h) = \sum_{i=1}^s b_i k_i(t, z, h)$ and $k_i(t, z, h) = f(t + c_i h, z + h(a_{i1} k_1(t, z, h) + \dots + a_{ii-1} k_{i-1}(t, z, h)))$.

We have

$$\|k_1(t, z_1, h) - k_1(t, z_2, h)\| = \|f(t + c_1 h, z_1) - f(t + c_1 h, z_2)\| \leq L \|z_1 - z_2\|,$$

$$\begin{aligned} \|k_2(t, z_1, h) - k_2(t, z_2, h)\| &= \|f(t + c_2 h, z_1 + h a_{21} k_1(t, z_1, h)) - f(t + c_2 h, z_2 + h a_{21} k_1(t, z_2, h))\| \\ &\leq L \|z_1 - z_2 + h a_{21} (k_1(t, z_1, h) - k_1(t, z_2, h))\| \leq L(1 + h |a_{21}| L) \|z_1 - z_2\| \end{aligned}$$

and so on. So we can conclude that

$$\begin{aligned} \|\Phi(t, z_1, h) - \Phi(t, z_2, h)\| &\leq \sum_{i=1}^s |b_i| \|k_i(t, z_1, h) - k_i(t, z_2, h)\| \\ &\leq L \left(\sum_{i=1}^s |b_i| + h L \sum_{i,j=1}^s |b_i a_{ij}| + h^2 L^2 \sum_{i,j,l=1}^s |b_i a_{ij} a_{jl}| + \dots \right) \|z_1 - z_2\| \\ &\leq L \left(\sum_{i=1}^s |b_i| + h_{max} L \sum_{i,j=1}^s |b_i a_{ij}| + h_{max}^2 L^2 \sum_{i,j,l=1}^s |b_i a_{ij} a_{jl}| + \dots \right) \|z_1 - z_2\|. \end{aligned}$$

Thus the result is proved with

$$\Lambda = L \left(\sum_{i=1}^s |b_i| + h_{max} L \sum_{i,j=1}^s |b_i a_{ij}| + h_{max}^2 L^2 \sum_{i,j,l=1}^s |b_i a_{ij} a_{jl}| + \dots \right).$$