Problem Sheet 3

September 27, 2024

Question 1

Consider the Kepler differential system you have implemented in Problem Sheet 1 :

$$\begin{aligned} \vec{x}''(t) &= \frac{-\vec{x}(t)}{\|\vec{x}(t)\|^3}, \quad 0 < t \leqslant T, \\ \vec{x}'(0) &= \vec{v}_0, \\ \vec{x}(0) &= \vec{x}_0, \end{aligned} \tag{1}$$

Given the same condition as in Problem Sheet 1, implement the RK4 method and compare the results to those obtained with the Euler scheme. You should provide a table like the one given in Answer Sheet 1.

Question 2

Let $y_0 \in \mathbb{R}^n$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be given, and let $y : \mathbb{R} \to \mathbb{R}$ be the solution of

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = y_0. \end{cases} \tag{2}$$

Let $Y(t) = \begin{pmatrix} t \\ y(t) \end{pmatrix} \in \mathbb{R}^{n+1}$ such that

$$\begin{cases} Y'(t) = F(Y(t)), \\ Y(0) = \begin{pmatrix} t_0 \\ y_0 \end{pmatrix}, \end{cases} \tag{3}$$

where $F(Y(t)) = \begin{pmatrix} 1 \\ f(t, y(t)) \end{pmatrix}$.

Apply a general s stages explicit RK method to (3) and check that it corresponds to an s stages explicit RK method for (2) provided

$$c_i = \sum_{j=1}^{i-1} a_{ij}, i = 1, ..., s \text{ and } \sum_{i=1}^{s} b_i = 1.$$

Question 3

Graded exercise for group 2

Consider the ordinary differential equation given by

$$\begin{cases} \dot{y}(t) = \lambda y(t), & 0 < t \leqslant T, \\ y(0) = y_0, \end{cases} \tag{4}$$

with $\lambda < 0$. Let N be a positive integer, let $h = \frac{T}{N}$ be the time step and $t_n = nh$ where $n = 0, 1, ..., N$. Consider now an order 4 RK scheme with 4 stages to approximate (4). Let

$$p_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

- Prove that $y_N = (p_4(\lambda h))^N y_0$.
- Prove that there exists $C > 0$ such that $\forall \lambda < 0, \forall h > 0$ such that $|p_4(\lambda h)| \leqslant 1$, $\forall T > 0$ and $\forall y_0 \in \mathbb{R}$,

$$|y(t_N) - y_N| \leqslant C|\lambda|^5 T h^4 |y_0|.$$

EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2024

Answer Key 3

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Question 1

The Runge-Kutta method shows an order four convergence. We can compare with forward Euler for $T = 40\pi$:

$h/2$	$\|\vec{e}^N\|$ explicit	$\|\vec{e}^N\|$ Runge-Kutta 4
0.0013	1.4062	$6.2701e-10$
0.0006	0.87651	$3.888e-11$
0.0003	0.49827	$2.3974e-12$
0.00015	0.26259	$4.6833e-13$
0.000075	0.13337	-
0.000033	0.06694	-

From the table, we can see that Forward Euler has a linear convergence, while RK4 has a 4th order convergence (note : the last two values in the table for RK4 are not reported because it has reached machine precision and the results are not meaningful).

Question 2

Let us write a general s stages RK method for (3) as

$$\begin{aligned}
K_1 &= F(Y(0)) \\
K_2 &= F(Y(0) + ha_{21}K_1) \\
K_3 &= F(Y(0) + h(a_{31}K_1 + a_{32}K_2)) \\
&\vdots \\
K_s &= F(Y(0) + h(a_{s1}K_1 + a_{s2}K_2 + ... + a_{ss-1}K_{s-1}) \\
Y_1 &= Y(0) + h(b_1K_1 + ... + b_sK_s)
\end{aligned}$$

By definition of F, we have that

$$K_1 = \begin{pmatrix} 1 \\ f(t_0, y_0) \end{pmatrix}.$$

We set $k_1 = f(t_0, y_0)$ and we have

$$K_1 = \begin{pmatrix} 1 \\ k_1 \end{pmatrix}.$$

Then, we have

$$Y_0 + ha_{21}K_1 = \begin{pmatrix} t_0 + ha_{21} \\ y_0 + ha_{21}k_1 \end{pmatrix}.$$

Thus

$$K_2 = \begin{pmatrix} 1 \\ f(t_0 + ha_{21}, y_0 + ha_{21}k_1) \end{pmatrix}.$$

Reproducing this process until the $s-th$ stages, we find that

$$K_s = \begin{pmatrix} 1 \\ k_s \end{pmatrix}$$

with $k_s = f(t_0 + h(a_{s1} + a_{s2} + ... + a_{ss-1}), y_0 + h(a_{s1}k_1 + ... + a_{ss-1}k_{s-1}))$.

Therefore we have that

$$Y_1 = \begin{pmatrix} t_0 \\ y_0 \end{pmatrix} + h\left(b_1 \begin{pmatrix} 1 \\ k_1 \end{pmatrix} + b_2 \begin{pmatrix} 1 \\ k_2 \end{pmatrix} + ... + b_s \begin{pmatrix} 1 \\ k_2 \end{pmatrix}\right) = \begin{pmatrix} t_0 + h(b_1 + ... + b_s) \\ y_0 + h(b_1 k_1 + ... + b_s k_s) \end{pmatrix}.$$

Thus, this is a s stages RK method provided

$$c_i = \sum_{j=1}^{i-1} a_{ij}, \quad 1 = \sum_{i=1}^{s} b_i.$$

Question 3

The exact solution of the ODE is :

$$y(t_N) = y_0 e^{\lambda N h}.$$

A 4 stages RK method applied to (4) yields :

$$\begin{aligned}
k_1 &= \lambda y_0 \\
k_2 &= \lambda y_0 + h\lambda^2 a_{21} y_0 \\
k_3 &= \lambda y_0 + h(a_{31} + a_{32})\lambda^2 y_0 + h^2 a_{32} a_{21} \lambda^3 y_0 \\
k_4 &= \lambda y_0 + h(a_{41} + a_{42} + a_{43})\lambda^2 y_0 + h^2(a_{42}a_{21} + a_{43}a_{31} + a_{43}a_{32})\lambda^3 y_0 + h^3 a_{43} a_{32} a_{21} \lambda^4 y_0
\end{aligned}$$

and

$$y_1 = y_0 + h(b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4).$$

Using explicit expressions of the k_i and the 4-th order conditions, we have that

$$y_1 = p_4(\lambda h) y_0.$$

By induction, we get

$$y_N = (p_4(\lambda h))^N y_0.$$

Then

$$y(t_N) - y_N = y_0 \left(e^{h\lambda N} - (p_4(\lambda h))^N\right) = y_0 \left(e^{\lambda h} - p_4(\lambda h)\right) \left(e^{\lambda h(N-1)} + e^{\lambda h(N-2)} p_4(\lambda h) + ... + (p_4(\lambda h))^{N-1}\right).$$

Assuming $p_4(\lambda h) \leqslant 1$ and since $|e^x - p_4(x)| \leqslant \frac{1}{5!}x^5 \ \forall x > 0$, we have

$$|y(t_N) - y_N| \leqslant \frac{1}{5!}|\lambda|^5 h^5 N |y_0|.$$

Below, we plot the stability domain of the method in the complex plane, i.e. for $\lambda \in \mathbb{C}$ the set S_{RK} defined by

$$S_{RK} = \{z = h\lambda : |p_4(z)| \leqslant 1\}.$$

We compare it to the stability domain of the forward Euler scheme

$$S_E = \{z = h\lambda : |1 + z| \leqslant 1\}.$$

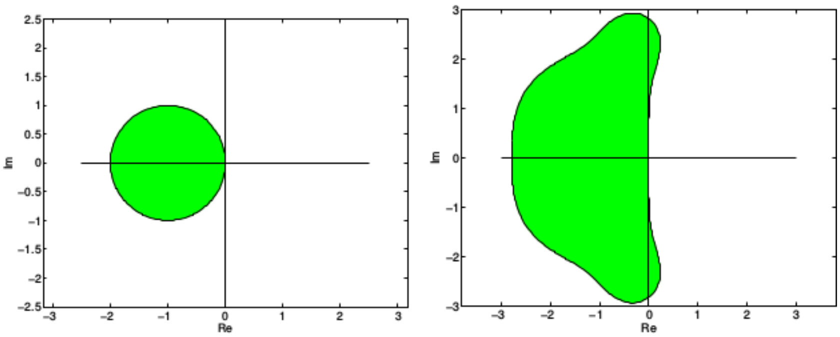

FIGURE 1 – Stability domain of the forward Euler scheme (left) and a 4-stages RK method of fourth order (right).