EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2024

Problem Sheet 2

September 20, 2024

Question 1

1. Taking the scalar product of the second equation of (1) with $\vec{V}(t)$ and using the first equation of (1) we can write :

$$\vec{V}(t)^T \vec{V}'(t) = -\vec{V}(t)^T A\vec{X}(t) = -\vec{X}'(t)^T A\vec{X}(t)$$

which can be expressed as follows

$$\frac{1}{2}\frac{d}{dt}\left(||\vec{V}(t)||^2 + \vec{X}(t)^T A\vec{X}(t)\right) = 0$$

we can obtain the result integrating between $t = 0$ and $t = T$.

2. Taking the scalar product of the second equation of (2) with $\left(\dfrac{\vec{V}_{n+1} + \vec{V}_n}{2}\right)$ and using the first equation (2) we can write :

$$\begin{aligned}\left(\frac{\vec{V}_{n+1} + \vec{V}_n}{2}\right)^T \frac{\vec{V}_{n+1} - \vec{V}_n}{h} &= -\left(\frac{\vec{V}_{n+1} + \vec{V}_n}{2}\right)^T A \quad \frac{\vec{X}_{n+1} + \vec{X}_n}{2} \\ &= -\left(\frac{\vec{X}_{n+1} - \vec{X}_n}{2}\right)^T A \quad \frac{\vec{X}_{n+1} + \vec{X}_n}{2}\end{aligned}$$

Thus we have

$$||\vec{V}_{n+1}||^2 + (\vec{X}_{n+1})^T A\vec{X}_{n+1} = ||\vec{V}_n||^2 + (\vec{X}_n)^T A\vec{X}_n.$$

which yields the result.

3. The difference of (2) with indices n and $n-1$ yields the Newmark scheme. $\vec{X}_1$ can be computed by writing :

$$\begin{cases} \dfrac{\vec{X}_1 - \vec{X}_0}{h} = \dfrac{\vec{V}_1 + \vec{V}_0}{2}, \\ \\ \vec{V}_1 = \vec{V}_0 - hA\dfrac{\vec{X}_1 + \vec{X}_0}{2} \end{cases} \tag{1}$$

yields

$$\left(\frac{h^2}{4}A + I\right)\vec{X}_1 = \left(h\vec{V}_0 + \left(\frac{h^2}{4}A + I\right)\vec{X}_0\right).$$