EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2024

Problem Sheet 1

September 09, 2024

Question 1

- We take the scalar product with $\vec{u}(t)$ and obtain $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{u}(t)\|^2 + \vec{u}(t)^T A\vec{u}(t) = \vec{u}(t)^T\vec{f}(t)$. We have $\vec{u}^T(t)A\vec{u}(t) \geqslant \lambda_{\min}\|\vec{u}(t)\|^2$. Using Cauchy-Schwarz and Young inequality ($ab \leqslant \frac{\lambda}{2}a^2 + \frac{1}{2\lambda}b^2$) we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{u}(t)\|^2 + \lambda_{\min}\|\vec{u}(t)\|^2 \leqslant \frac{\lambda_{\min}}{2}\|\vec{u}(t)\|^2 + \frac{1}{2\lambda_{\min}}\|\vec{f}(t)\|^2,$$

which yields the result integrating from $t = 0$ to $t = T$.

- From Taylor expansion we find $\vec{u}(t_n) = \vec{u}(t_{n+1}) - h\vec{u}'(t_{n+1}) + \frac{h^2}{2}\vec{u}''(s_n)$, with $t_n < s_n < t_{n+1}$. This yields

$$\frac{\vec{u}(t_{n+1}) - \vec{u}(t_n)}{h} + A\vec{u}(t_{n+1}) = \vec{f}(t_{n+1}) - \frac{h}{2}\vec{u}''(s_n),$$

hence $\vec{r}^{n+1} = -\frac{h}{2}\vec{u}''(s_n)$ and $\|\vec{r}^{n+1}\| \leqslant \frac{1}{2}h \max_{0\leqslant t\leqslant T}\|\vec{u}''(t)\|$.

- Subtracting (2) and (3) and taking the scalar product with $\vec{e}^{n+1}$we find $\frac{1}{h}\left(\vec{e}^{n+1} - \vec{e}^{n}\right)^T \vec{e}^{n+1} + (\vec{e}^{n+1})^T A\vec{e}^{n+1} = (\vec{e}^{n+1})^T\vec{r}^{n+1}$. Thus, $\frac{1}{2h}\left(\|\vec{e}^{n+1}\|^2 - \|\vec{e}^{n}\|^2 + \|\vec{e}^{n+1} - \vec{e}^{n}\|^2\right) + \lambda_{\min}\|\vec{e}^{n+1}\|^2 \leqslant \frac{\lambda_{\min}}{2}\left\|\vec{e}^{n+1}\right\|^2 + \frac{1}{2\lambda_{\min}}\left\|\vec{r}^{n+1}\right\|^2$, which yields the result.
- Summing from $n = 0$ to $N - 1$ and since $\vec{e}^0 = 0$ we find

$$\frac{1}{h}\|\vec{e}^N\|^2 + \lambda_{\min}\sum_{n=0}^{N-1}\|\vec{e}^{n+1}\|^2 \leqslant \sum_{n=0}^{N-1}\frac{1}{\lambda_{\min}}\|\vec{r}^{n+1}\|^2 \leqslant \frac{NC^2h^2}{\lambda_{\min}}.$$

From $h = \frac{T}{N}$ we obtain our result.

Question 2

(a) We have $\vec{u}(t_{n+1}) = \vec{u}(t_n) + h\vec{u}'(t_n) + \frac{1}{2}h^2\vec{u}''(s_n), \quad t_n < s_n < t_{n+1}$. Thus

$$\frac{\vec{u}(t_{n+1}) - \vec{u}(t_n)}{h} + A\vec{u}(t_n) = \vec{f}(t_n) + \vec{r}^n,$$

with $\vec{r}^n = \frac{1}{2}h\vec{u}''(s_n)$ and $\|\vec{r}^n\| \leqslant \frac{1}{2}h\max_{0\leqslant t\leqslant T}\|\vec{u}''(t)\|$. Defining $\vec{e}^n = \vec{u}(t_n) - \vec{u}^n$, we have

$$\frac{\vec{e}^{n+1} - \vec{e}^n}{h} + A\vec{e}^n = \vec{r}^n.$$

Multiplying the result by $\vec{e}^{n+1}$ it yields

$$\frac{1}{h}\left(\vec{e}^{n+1} - \vec{e}\right)^T \vec{e}^{n+1} + (\vec{e}^{n+1})^T A\vec{e}^n = (\vec{e}^{n+1})^T\vec{r}^n,$$

which yields the result, using $a = (a - b) + b$ and $(a - b)a = \frac{1}{2}(a^2 - b^2 + (a - b)^2)$.

(b) Defining the scalar product $< \vec{u}, \vec{v} >_A = \vec{u}^T A\vec{v}$, we use the Cauchy-Schwarz and Young inequality to find $< \vec{u}, \vec{v} >_A \leqslant \|\vec{u}\|_A \|\vec{v}\|_A \leqslant \frac{1}{2}\|\vec{u}\|_A^2 + \frac{1}{2}\|\vec{v}\|_A^2$. This yields the result choosing $\vec{u} = \vec{e}^{n+1}$ and $\vec{v} = \vec{e}^{n+1} - \vec{e}^n$.

(c) From (a) and (b), we have

$$\frac{1}{2h}\left(\left\|\vec{e}^{n+1}\right\|^2 - \|\vec{e}^{n}\|^2 + \left\|\vec{e}^{n+1}-\vec{e}^{n}\right\|^2\right) + \frac{1}{2}(\vec{e}^{n+1})^T A\vec{e}^{n+1} \leqslant (\vec{e}^{n+1})^T\vec{r}^{n} + \frac{1}{2}(\vec{e}^{n+1}-\vec{e}^{n})^T A(\vec{e}^{n+1}-\vec{e}^{n}).$$

The result is obtained since $\lambda_{\min}\|\vec{v}\|^2 \leqslant \vec{v}^T A\vec{v} \leqslant \lambda_{\max}\|\vec{v}\|^2$, $\forall\vec{v}\in\mathbb{R}^d$ (from decomposition into orthonormal basis of eigenvectors of A).

(d) We have

$$\begin{aligned}\left(\frac{1}{2h}+\frac{\lambda_{\min}}{2}\right)\left\|\vec{e}^{n+1}\right\|^2 + \left(\frac{1}{2h}-\frac{\lambda_{\max}}{2}\right)\left\|\vec{e}^{n+1}-\vec{e}^{n}\right\|^2 &\leqslant \frac{1}{2h}\|\vec{e}^{n}\|^2 + \left\|\vec{e}^{n+1}\right\|\|\vec{r}^{n}\| \\ &\leqslant \frac{1}{2h}\|\vec{e}^{n}\|^2 + \frac{\lambda_{\min}}{2}\left\|\vec{e}^{n+1}\right\|^2 + \frac{1}{2\lambda_{\min}}\|\vec{r}^{n}\|^2.\end{aligned}$$

If $\frac{1}{2h}-\frac{\lambda_{\max}}{2}\geqslant 0$, this yields $\left\|\vec{e}^{n+1}\right\|^2 \leqslant \|\vec{e}^{n}\|^2 + \frac{h}{\lambda_{\min}}\|\vec{r}^{n}\|^2 \leqslant \|\vec{e}^{n}\|^2 + \frac{C^2h^3}{\lambda_{\min}}$. Summing from $n=0$ to $n=N-1$ and using $T=Nh$ yields the result.

Question 3

We have from (4) :

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{x}'(t)\|^2 = \vec{x}''(t)^T\vec{x}'(t) = \frac{-\vec{x}(t)^T\vec{x}'}{\|\vec{x}(t)\|^3} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\|\vec{x}(t)\|}.$$

Now let $\vec{x}'(t)=\vec{v}(t)$, (1) is equivalent to $\vec{u}'(t)=f(\vec{u}(t))$, with $\vec{u}(t)=\begin{pmatrix}\vec{x}(t)\\ \vec{v}(t)\end{pmatrix}$ and $f(\vec{x},\vec{v})=\begin{pmatrix}\vec{v}\\ \frac{-\vec{x}}{\|\vec{x}\|^3}\end{pmatrix}$. Let N be a positive integer, $h=\frac{T}{N}, t_n=nh, n=0,1,\dots,N$. We compute $\vec{u}^{n+1}\in\mathbb{R}^d, n=0,1,\dots,N-1$, with the Euler explicit scheme :

$$\frac{\vec{u}^{n+1}-\vec{u}^{n}}{h} = \vec{f}(\vec{u}^{n}).$$

Let $\vec{e}$ be the error at final time, the following results are obtained :

h	$\|\vec{e}\|$ explicit
0.0002	0.330
0.0001	0.169
0.00005	0.085
0.000025	0.042

which shows that the error is $O(h)$.

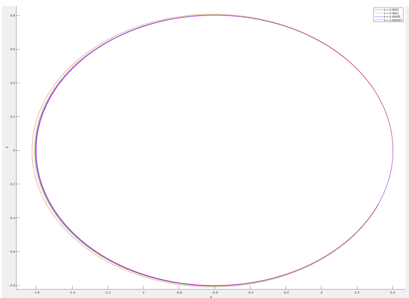

Figure 1 – Numerical solution with the Forward Euler method for the different hs.

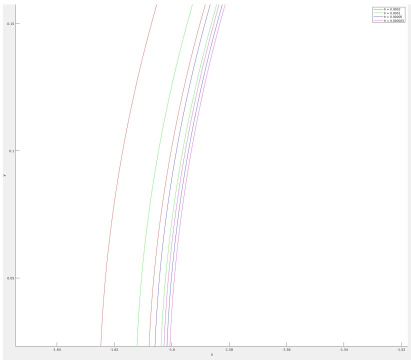

Figure 2 – Zoom of the left side of the previous plot.