EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2024

Answer Key 13

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Question 1

(a) First, observe that, by eliminating the x variable, we can set that

$$BA^{-1}B^T\mu^* = BA^{-1}b - c.$$

Then we have, using that $x^k = A^{-1}b - A^{-1}B^T\mu^k$,

$$\begin{aligned}\mu^* - \mu^{k+1} &= \mu^* - \mu^k - \rho(Bx^k - c)\\ &= \mu^* - \mu^k - \rho(BA^{-1}b - BA^{-1}B^T\mu^k - c)\\ &= \mu^* - \mu^k - \rho BA^{-1}B^T(\mu^* - \mu^k)\\ &= (I - \rho BA^{-1}B^T)(\mu^* - \mu^k).\end{aligned}$$

- Let $\varphi_j, j = 1, ..., p$ be a orthonormal basis of eigenvectors of $BA^{-1}B^T$. Let us expand $\mu^* - \mu^0$ in the basis :

$$\mu^* - \mu^0 = \sum_{j=1}^{p} \alpha_j\varphi_j.$$

Then, we have

$$(I - \rho BA^{-1}B^T)(\mu^* - \mu^0) = \sum_{j=1}^{p}(1 - \rho\lambda_j)\alpha_j\varphi_j,$$

and by induction, using what we proved before :

$$\mu^* - \mu^k = \sum_{j=1}^{p}(1 - \rho\lambda_j)^k\alpha_j\varphi_j.$$

Then, since the φ_j form a orthonormal basis, we have

$$\|\mu^* - \mu^k\|^2 = \sum_{j=1}^{p}(1 - \rho\lambda_j)^{2k}\alpha_j^2 \leqslant \max_{j=1,..,p} |1 - \rho\lambda_j|^{2k}\|\mu^* - \mu^0\|^2.$$

Then it follows that $\|\mu^* - \mu^k\| \to 0$ as $k \to \infty$ if

$$|1 - \rho\lambda_j| < 1, \forall\, j = 1, ..., p$$

which is satisfied if $\rho < {}^2\!/\!_{\lambda_p}$.

(b) Let $\mu \in \mathbb{R}^p$ and choose $y \in \mathbb{R}^n$ such that $Ay = B^T\mu$. Let $x \in \mathbb{R}^n$, we have

$$\frac{(x^TB^T\mu)^2}{x^TAx} = \frac{(x^TAy)^2}{x^TAx} \leqslant \frac{(x^TAx)(y^TAy)}{x^TAx} \leqslant y^TAy,$$

where we used the Cauchy-Schwarz inequality for the inner product y^TAx. We have proved that for all $x \in \mathbb{R}^n$

$$\frac{(x^TB^T\mu)^2}{x^TAx} \leqslant y^TAy.$$

with equality if $x = y$. Thus,

$$\max_{x\in\mathbb{R}^n x\neq 0} \frac{(x^TB^T\mu)^2}{x^TAx} = y^TAy = \mu^TBA^{-1}B^T\mu.$$

(c) Using (1) and point (ii), we have

$$C_1^2\|\mu\|^2 \leqslant \mu^T BA^{-1}B^T\mu \leqslant C_2^2\|\mu\|^2.$$

Choosing $\mu = \varphi_j$ yields

$$C_1^2 \leqslant \lambda_j \leqslant C_2^2,$$

and therefore we get

$$1-\rho C_2^2 \leqslant 1-\rho\lambda_j \leqslant 1-\rho C_1^2.$$

Then $|1-\rho\lambda_j| < 1$ if $\rho < 2/C_2^2$ and $\rho < 2/C_1^2$. Since $C_2 \geqslant C_1$, it is sufficient to require that $\rho < 2/C_2$.

Question 2

Multiply (2) by u_i, sum over the indices $i = 1,2,3$ and integrate over Ω :

$$\int_\Omega \sum_{i=1}^3 \left(\rho\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^3 u_j \frac{\partial}{\partial x_j} u_i\right) u_i - \mu\Delta u_i\ u_i + \frac{\partial p}{\partial x_i} u_i\right)\ \mathrm{d}\vec{x} = 0. \tag{*}$$

The first term of (*) is equal to

$$\int_\Omega \sum_{i=1}^3 \rho \frac{\partial u_i}{\partial t} u_i\ \mathrm{d}x = \int_\Omega \sum_{i=1}^3 \frac{1}{2}\rho\frac{\partial}{\partial t}\left(u_i^2\right)\ \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\rho\left\|\vec{u}(\vec{x},t)\right\|^2\ \mathrm{d}\vec{x}.$$

For $f, g : \Omega \to \mathbb{R}$, the following formula holds :

$$\int_\Omega \frac{\partial}{\partial x_i}(fg)\ \mathrm{d}\vec{x} = \int_{\partial\Omega} fgn_i\ \mathrm{d}s = \int_\Omega g\frac{\partial}{\partial x_i}f + f\frac{\partial}{\partial x_i}g\ \mathrm{d}\vec{x},$$

where $\vec{n}(\vec{x}) = (n_1,n_2,n_3)(\vec{x})$ denotes the unitary vector normal to the boundary $\vec{x} \in \partial\Omega$. Hence using (3), we have :

$$\begin{aligned}
&\int_\Omega \left(u_1\frac{\partial}{\partial x_1}u_1 + u_2\frac{\partial}{\partial x_2}u_1 + u_3\frac{\partial}{\partial x_3}u_1\right) u_1\ \mathrm{d}\vec{x}\\
&= -\int_\Omega \left(\frac{\partial}{\partial x_1}(u_1^2) + \frac{\partial}{\partial x_2}(u_1u_2) + \frac{\partial}{\partial x_3}(u_1u_3)\right) u_1\ \mathrm{d}\vec{x} + \int_\Omega \left(\frac{\partial}{\partial x_1}(u_1^3) + \frac{\partial}{\partial x_2}(u_1^2u_2) + \frac{\partial}{\partial x_3}(u_1^2u_3)\right)\ \mathrm{d}\vec{x}\\
&= -\int_\Omega \left(\frac{\partial}{\partial x_1}(u_1^2) + \frac{\partial}{\partial x_2}(u_1u_2) + \frac{\partial}{\partial x_3}(u_1u_3)\right) u_1\ \mathrm{d}\vec{x} + \underbrace{\int_{\partial\Omega} u_1^2(u_1n_1 + u_2n_2 + u_3n_3)\ \mathrm{d}s}_{=0 \text{ since } u_i=0 \text{ on } \partial\Omega}\\
&= \int_\Omega u_1\left(u_1\underbrace{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)}_{=0} + \left(u_1\frac{\partial}{\partial x_1}u_1 + u_2\frac{\partial}{\partial x_2}u_1 + u_3\frac{\partial}{\partial x_3}u_1\right)\right)\ \mathrm{d}\vec{x}.
\end{aligned}$$

Thus,

$$2\int_\Omega \left(u_1\frac{\partial}{\partial x_1}u_1 + u_2\frac{\partial}{\partial x_2}u_1 + u_3\frac{\partial}{\partial x_3}u_1\right) u_1\ \mathrm{d}\vec{x} = 0.$$

The third term of (*) equals

$$\sum_{i=1}^3 \int_\Omega \mu\Delta u_i\ u_i\ \mathrm{d}\vec{x} = \sum_{i=1}^3 \int_\Omega \mu\left\|\nabla u_i(\vec{x},t)\right\|^2\ \mathrm{d}\vec{x} + \sum_{i=1}^3 \underbrace{\int_{\partial\Omega} \mu u_i(\nabla u_i\cdot\vec{n})\ \mathrm{d}s}_{=0 \text{ since } u_i=0 \text{ on } \partial\Omega}.$$

The last two terms of (*) cancel since

$$\int_\Omega \frac{\partial}{\partial x_i}(pu_i)\ \mathrm{d}\vec{x} = \int_\Omega u_i\frac{\partial}{\partial x_i}p\ \mathrm{d}\vec{x} + \int_\Omega p\frac{\partial}{\partial x_i}u_i\ \mathrm{d}\vec{x} = \underbrace{\int_{\partial\Omega} pu_in_i\ \mathrm{d}s}_{=0} \text{ since } u_i = 0 \text{ on } \partial\Omega.$$