EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2024

Answer Key 12

December 2, 2024

Question 1

(a) Do the scalar product of the schema with $\vec{u}^{n+1} - \vec{u}^{n-1}$, then :

$$\left\| \frac{\vec{u}^{n+1} - \vec{u}^n}{\tau} \right\|^2 + \left(\frac{\vec{u}^{n+1} + \vec{u}^n}{2} \right)^T A \left(\frac{\vec{u}^{n+1} + \vec{u}^n}{2} \right) - \left\| \frac{\vec{u}^n - \vec{u}^{n-1}}{\tau} \right\|^2$$
$$- \left(\frac{\vec{u}^n + \vec{u}^{n-1}}{2} \right)^T A \left(\frac{\vec{u}^n + \vec{u}^{n-1}}{2} \right) = \tau (\vec{f}^n)^T \left(\frac{\vec{u}^{n+1} - \vec{u}^n}{\tau} + \frac{\vec{u}^n - \vec{u}^{n-1}}{\tau} \right)$$

Set $y^{n+1} = \left(\left\| \frac{\vec{u}^{n+1} - \vec{u}^n}{\tau} \right\|^2 + \left(\frac{\vec{u}^{n+1} + \vec{u}^n}{2} \right)^T A \left(\frac{\vec{u}^{n+1} + \vec{u}^n}{2} \right) \right)^{\frac{1}{2}}$ we get,

$$(y^{n+1})^2 - (y^n)^2 \leqslant C^n \tau (y^{n+1} + y^n)$$

with $C^n = ||\vec{f}^n||$. Then, since $a^2 - b^2 = (a-b)(a+b)$, we get,

$$(y^{n+1} - y^n) \leqslant C^n \tau$$

and finally iterating in time,

$$y^M \leqslant y^0 + \tau \sum_{n=0}^{M-1} C^n$$

(b) Starting from the wave equation $\frac{\partial^2 u}{\partial t^2}(x_i, t_n) - c^2 \frac{\partial^2 u}{\partial x^2}(x_i, t_n) = f(x_i, t_n)$ we have to retrieve the implicit discretization of the wave equation. To do this, since $u \in C^4([0,1] \times [0,T])$, we can rewrite the equation has :

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_n) - c^2 \frac{\frac{\partial^2 u}{\partial x^2}(x_i, t_{n+1}) + 2\frac{\partial^2 u}{\partial x^2}(x_i, t_n) + \frac{\partial^2 u}{\partial x^2}(x_i, t_{n-1})}{4} = f(x_i, t_n) + \alpha_i^n$$

where $\alpha_i^n \leqslant C\tau^2$ and comes from the following steps :

$$u(t_{n-1}) = v(t_n) - \tau v'(t_n) + \frac{\tau^2}{2} v''(\beta_n),$$

$$u(t_{n+1}) = v(t_n) + \tau v'(t_n) + \frac{\tau^2}{2} v''(\gamma_n),$$

then,

$$\frac{u(t_{n-1}) + 2u(t_n) + u(t_{n+1})}{4} = v(t_n) + \frac{\tau^2}{2}(v''(\beta_n) + v''(\gamma_n)).$$

Then, you use a centered differences schema to discretize in space and in time as usual and get to the conclusion that $|r_i^n| \leqslant C(h^2 + \tau^2)$. The final schema is written as,

$$\frac{\vec{U}^{n-1} - 2\vec{U}^n + \vec{U}^{n+1}}{\tau^2} + A \frac{\vec{U}^{n-1} + 2\vec{U}^n + \vec{U}^{n+1}}{4} = \vec{f}^n + \vec{r}^n.$$

(c) Finally, by writing $\vec{E}^n = \vec{U}^n - \vec{u}^n$, you get,

$$\frac{\vec{E}^{n-1} - 2\vec{E}^n + \vec{E}^{n+1}}{\tau^2} + A \frac{\vec{E}^{n-1} + 2\vec{E}^n + \vec{E}^{n+1}}{4} = \vec{r}^n.$$

proceding as in point one : multiply everithing by $\vec{E}^{n+1} - \vec{E}^{n-1}$, set $z^{n+1} = \left(\left\| \frac{\vec{E}^{n+1} - \vec{E}^n}{\tau} \right\|^2 + \left(\frac{\vec{E}^{n+1} + \vec{E}^n}{2} \right)^T A \left(\frac{\vec{E}^{n+1} + \vec{E}^n}{2} \right) \right)$ and iterate in time,

$$z^M \leqslant z^1 + \tau \sum_{n=0}^{M-1} ||r^n|| \leqslant z^0 + \frac{M\tau C}{\sqrt{h}}(h^2 + \tau^2).$$

Question 2

(a) See wave.m

(b) As seen in the lecture, $e_{inf} := \max_{1 \leqslant i \leqslant N} |u_i^M - u(x_i, t_M)| = \mathcal{O}(h^2)$ when $h = \tau$:

$N+1$	M	e_{inf}
50	50	$1.97 \cdot 10^{-3}$
100	100	$4.93 \cdot 10^{-4}$
200	200	$1.23 \cdot 10^{-4}$
400	400	$3.08 \cdot 10^{-5}$
800	800	$7.71 \cdot 10^{-6}$

(c) When $N = 199$ and $M = 199$, the scheme becomes unstable, as shown in the figure below.

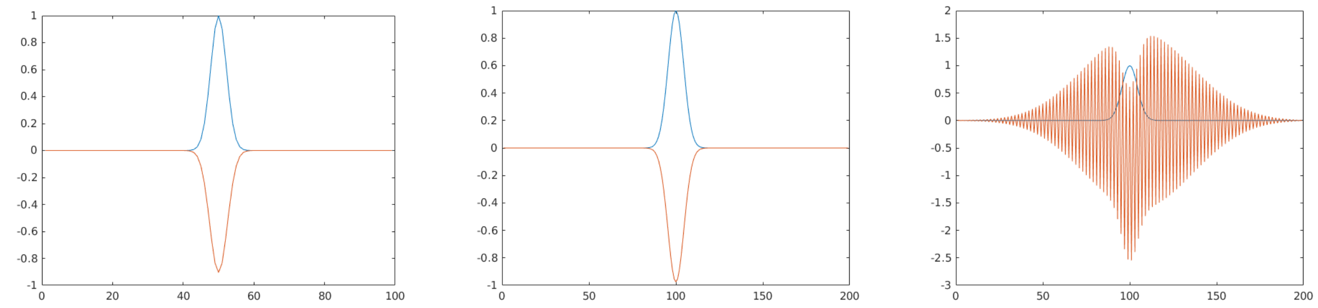

FIGURE 1 – u_0 (blue) and u^M (orange) for various values of N and M. Left : $N = 99$, $M = 100$. Center : $N = 199$, $M = 200$. Right : $N = 199$, $M = 199$.