EPFL - Advanced Numerical Analysis - Marco Picasso - Spring 2024

Answer Key 11

November 29, 2024

Question 1

(a) The exact solution is discontinuos, it is 0 for $t < 0.5$ and 1 for $t > 0.5$

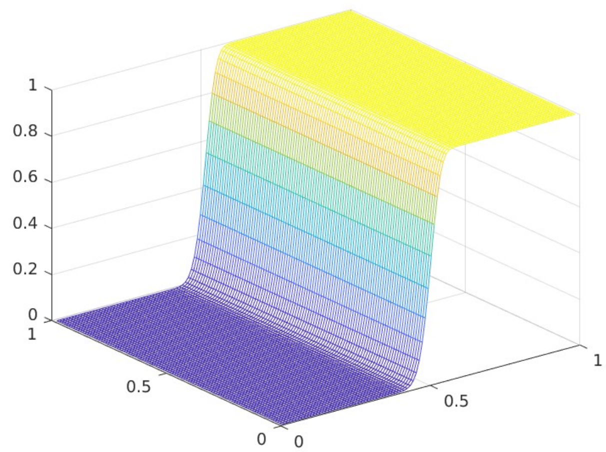

FIGURE 1 – Solution of the trasport problem for N=99 and M=60.

(b) The lines of the code are :

— Line 7 : E = sparse(2 :L,1 :L-1,1,L,L) ;

— Line 8 : A = kron(I,I-E)*(L+1) ;

— Line 9 : mat=speye(L*L,L*L)-tau*A ;

— Line 12 : u=mat*u ;

You can observe that letting $h \to 0$ and $\tau \to 0$ the solution at the discontinuity becomes steeper and steeper getting ever closer to the exact solution.

(c) For $N = 99$ and $M = 50$ the CFL number is equal to 1. This lead to have the exact solution since $h = \tau$ and there is a node in $t = 0.5$:

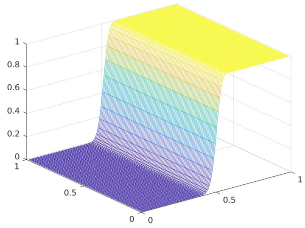

FIGURE 2 – Solution of the trasport problem for N=99 and M=60, CFL = 0.8333.

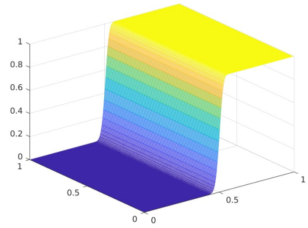

FIGURE 3 – Solution of the trasport problem for N=199 and M=120, CFL = 0.8333.

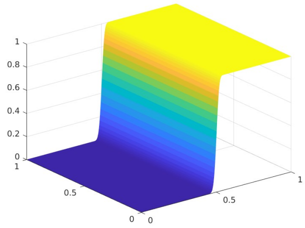

FIGURE 4 – Solution of the trasport problem for N=399 and M=240, CFL = 0.8333.

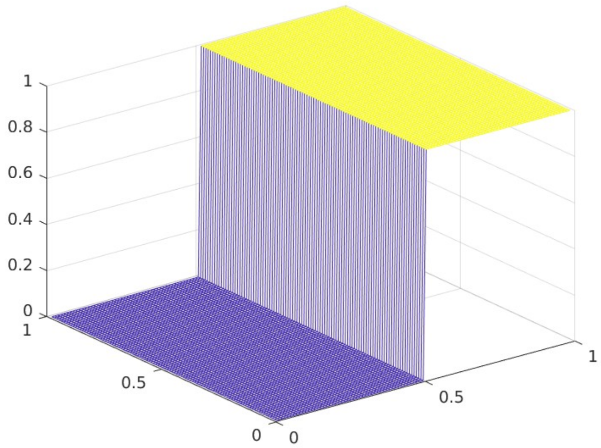

FIGURE 5 – Solution of the trasport problem for N=99 and M=50, CFL=1.

Question 2

(a) We assume that $\sigma \in \mathcal{C}^2([0,1] \times [0,T])$. Then, we have

$$\frac{\partial^2 u}{\partial t^2}(x,t) - c^2 \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 \sigma}{\partial t \partial x}(x,t) - c^2 \frac{\partial^2 \sigma}{\partial x \partial t}(x,t) = 0$$

and

$$\frac{\partial u}{\partial t}(x,0) = \frac{\partial \sigma}{\partial x}(x,0) = \sigma_0'(x).$$

(b) The Newmark's scheme reads

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2} - c^2 \frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{4\ h^2} - 2\ c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{4\ h^2} - c^2 \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{4\ h^2} = 0.$$

If u and σ satisfy the Stormer-Verlet scheme, then

$$\begin{aligned}
&\frac{u_i^{n+1}-2u_i^n+u_i^{n-1}}{\tau^2}-c^2\frac{u_{i+1}^{n-1}-2u_i^{n-1}+u_{i-1}^{n-1}}{4\ h^2}-2\ c^2\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{4\ h^2}-c^2\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{4\ h^2}=\\
&=\frac{1}{2}\frac{\sigma_{i+\frac{1}{2}}^{n+1}-\sigma_{i-\frac{1}{2}}^{n+1}}{\tau h}+\frac{1}{2}\frac{\sigma_{i+\frac{1}{2}}^{n}-\sigma_{i-\frac{1}{2}}^{n}}{\tau h}-\frac{1}{2}\frac{\sigma_{i+\frac{1}{2}}^{n}-\sigma_{i-\frac{1}{2}}^{n}}{\tau h}-\frac{1}{2}\frac{\sigma_{i+\frac{1}{2}}^{n-1}-\sigma_{i-\frac{1}{2}}^{n-1}}{\tau h}\\
&-\frac{1}{2}\frac{\sigma_{i+\frac{1}{2}}^{n+1}-\sigma_{i+\frac{1}{2}}^{n}}{\tau h}+\frac{1}{2}\frac{\sigma_{i-\frac{1}{2}}^{n+1}-\sigma_{i-\frac{1}{2}}^{n}}{\tau h}-\frac{1}{2}\frac{\sigma_{i+\frac{1}{2}}^{n}-\sigma_{i+\frac{1}{2}}^{n-1}}{\tau h}+\frac{1}{2}\frac{\sigma_{i-\frac{1}{2}}^{n}-\sigma_{i-\frac{1}{2}}^{n-1}}{\tau h}=0
\end{aligned}$$

and thus u indeed satisfies the Newmark's scheme.