EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2024

Answer Key 10

November 18, 2024

Question 1

(a) Let k be such that $|u_k^{n+1}| \geqslant |u_j^{n+1}|$, $1 \leqslant j \leqslant N$. If $k = 1$, we set $u_{k-1}^n = u_{k-1}^{n+1} = 0$; if $k = N$, we set $u_{k+1}^n = u_{k+1}^{n+1} = 0$.

Then

$$\left(1 + \frac{\tau}{h^2}\right) u_k^{n+1} = \left(1 - \frac{\tau}{h^2}\right) u_k^n + \frac{\tau}{2h^2}(u_{k-1}^n + u_{k+1}^n) + \frac{\tau}{2h^2}(u_{k-1}^{n+1} + u_{k+1}^{n+1}).$$

Therefore, if $\tau \leqslant h^2$,

$$\begin{aligned}\left(1 + \frac{\tau}{h^2}\right) |u_k^{n+1}| &\leqslant \left(1 - \frac{\tau}{h^2}\right) |u_k^n| + \frac{\tau}{2h^2}(|u_{k-1}^n| + |u_{k+1}^n|) + \frac{\tau}{2h^2}(|u_{k-1}^{n+1}| + |u_{k+1}^{n+1}|) \\ &\leqslant \left(1 - \frac{\tau}{h^2}\right) |u_k^n| + \frac{\tau}{2h^2}(|u_{k-1}^n| + |u_{k+1}^n| + \frac{\tau}{2h^2}(|u_k^{n+1}| + |u_k^{n+1}|)\end{aligned}$$

and thus

$$\begin{aligned}|u_k^{n+1}| &\leqslant \left(1 - \frac{\tau}{h^2}\right) |u_k^n| + \frac{\tau}{2h^2}(|u_{k-1}^n| + |u_{k+1}^n|) \\ &\leqslant ||u^n||_\infty.\end{aligned}$$

(b)

$$\begin{aligned}||\vec{u}^{n+1}||_2 &= \left|\left|\left(I + \frac{\tau A}{2}\right)^{-1} \left(I - \frac{\tau A}{2}\right) \vec{u}^n\right|\right|_2 \\ &\leqslant \max_{1 \leqslant i \leqslant N} \frac{1 - \frac{\tau \lambda_i}{2}}{1 + \frac{\tau \lambda_i}{2}} ||\vec{u}^n||_2 \\ &\leqslant ||\vec{u}^n||_2,\end{aligned}$$

where λ_i, $i = 1, \ldots, N$ are the eigenvalues of the matrix A.

Question 2

We have

$$\int_0^1 \frac{\partial u}{\partial t} u^- dx + \int_0^1 \frac{\partial u}{\partial x} \frac{\partial u^-}{\partial x} dx = \int_0^1 f u^- dx$$

and thus

$$-\frac{d}{dt} \int_0^1 (u^-)^2 dx - \int_0^1 \left(\frac{\partial u^-}{\partial x}\right)^2 dx = \int_0^1 f u^- dx.$$

Therefore, for every $t > 0$,

$$\frac{1}{2} \int_0^1 \left(u^-(x,t)\right)^2 + \underbrace{\int_0^t \int_0^1 \left(\frac{\partial u^-}{\partial x}(x,t)\right)^2 dxdt}_{\geqslant 0} = \underbrace{- \int_0^t \int_0^1 \underbrace{f(x,t) u^-(x,t)}_{\geqslant 0} dxdt}_{\leqslant 0} + \frac{1}{2} \int_0^1 \underbrace{(u_0^-(x))^2}_{0} dx.$$

Thus,

$$\int_0^1 (u^-(x,t))^2 dx = 0$$

so that $u^-(x,t) = 0$ and therefore $u(x,t) \geqslant 0$.

Question 3

Numerical approximation of the advection equation for $N = 99$ and $M = 80$, so that $0.006 = \tau \leqslant \frac{h}{|c_0|} = 0.007$:

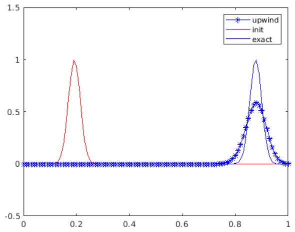

FIGURE 1 – Upwind scheme.

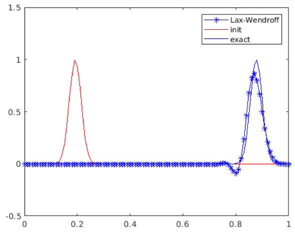

FIGURE 2 – Lax-Wendroff scheme.

For the Upwind scheme :

N	M	$\mathrm{err}_{\mathrm{inf}}$
49	40	$5.58 \cdot 10^{-1}$
99	80	$4.26 \cdot 10^{-1}$
199	160	$2.96 \cdot 10^{-1}$
399	320	$1.86 \cdot 10^{-1}$
799	640	$1.08 \cdot 10^{-1}$

For the Lax-Wendroff scheme :

N	M	$\mathrm{err}_{\mathrm{inf}}$
49	40	$3.96 \cdot 10^{-1}$
99	80	$2.23 \cdot 10^{-1}$
199	160	$9.09 \cdot 10^{-2}$
399	320	$2.45 \cdot 10^{-2}$
799	640	$6.13 \cdot 10^{-3}$

Question 4

(a) We show it by induction. The case $n = 0$ is trivially true. We assume that the property holds true for n and prove it for $n + 1$.
We have

$$\begin{aligned} u_j^{n+1} &= u_j^0 \gamma^n - \frac{\alpha}{2}(u_{j+1}^0 \gamma^n - u_{j-1}^0 \gamma^n) + \frac{\tau^2 c_0^2}{2h^2}(u_{j-1}^0 \gamma^n - 2u_j^0 \gamma^n + u_{j+1}^0 \gamma^n) \\ &= \gamma^n e^{imjh}(1 - \alpha \frac{e^{imh} - e^{-imh}}{2} - \alpha^2 + \alpha^2 \frac{e^{imh} + e^{-imh}}{2} \\ &= e^{imjh}\gamma^n(1 - \alpha^2(1 - \cos(mh)) - i\alpha \sin(mh)) \\ &= e^{imjh}\gamma^n \gamma. \end{aligned} \tag{1}$$

(b)

$$\begin{aligned} |\gamma|^2 &= 1 - \alpha^2(1 - \cos(mh))^2 + \alpha^2 \sin^2(mh) \\ &= 1 - 2\alpha^2(1 - \cos(mh)) + \alpha^4(1 - \cos(mh))^2 + \alpha^2(1 - \cos^2(mh)) \\ &= \alpha^4(1 - \cos(mh))^2 - \alpha^2(1 - \cos(mh))^2 + 1 \\ &= 1 + \alpha^2(\alpha^2 - 1)(1 - \cos(mh))^2. \end{aligned} \tag{2}$$