- Let Y be a random vector in $\mathbb{R}^d$ with covariance matrix Ω.
- Find direction $v_1 \in \mathbb{S}^{d-1}$ such that the projection of Y onto v_1 has maximal variance.
- For $j = 2, 3, \ldots, d$, find direction $v_j \perp v_{j-1}$ such that projection of Y onto v_j has maximal variance.

Solution: maximise $\mathrm{Var}(v_1^\top Y) = v_1^\top \Omega v_1$ over $\|v_1\| = 1$

$$v_1^\top \Omega v_1 = v_1^\top U \Lambda U^\top v_1 = \|\Lambda^{1/2} U^\top v_1\|^2 = \sum_{i=1}^{d} \lambda_i (u_i^\top v_1)^2 \qquad \text{[change of basis]}$$

$$v_1 = \sum \langle v_1, u_i \rangle u_i \Rightarrow \sum \langle v_1, u_i \rangle e_i = U^T v_1$$

Now $\sum_{i=1}^d (u_i^\top v_1)^2 = \|v_1\|^2 = 1$ so we have a convex combination of the $\{\lambda_j\}_{j=1}^d$,

$$\sum_{i=1}^{d} p_i \lambda_i, \qquad \sum_i p_i = 1, \quad p_i \geq 0, \quad i = 1, \ldots, d.$$

But $\lambda_1 \geq \lambda_i \geq 0$ so clearly this sum is maximised when $p_1 = 1$ and $p_j = 0$ $\forall j \neq 1$, i.e. $v_1 = \pm u_1$.

Iteratively, $v_j = \pm u_j$, i.e. principal components are eigenvectors of Ω.