MATH-329 Nonlinear optimization
Exercise session 7: Intuitive constrained optimization

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1. Warm-up: drawing tangent and normal cones. Consider the following sets:

1. The half disk $\{x \in \mathbb{R}^2 : \|x\| \leq 1 \text{ and } x_1 \geq 0\}$.
2. The "opposite" of the half disk, $\mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 : \|x\| < 1 \text{ and } x_1 > 0\}$.
3. The triangle limited by the vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.

For each of these, do the following. (Since this is a warm-up exercise, no proofs needed: just draw.)

1. Draw the set.
2. Identify interesting points, and draw the tangent cones there.
3. At the same points, draw also the normal cones.

Consider the definition of stationary points for minimization problems constrained to those sets, specifically the formulation that relates gradients and normal cones. Make sure this concept makes sense to you.

2. Necessary optimality conditions in the interior of the constraint set. Consider a set S and a point x in the interior of S (that is, there exists a neighborhood of x which is entirely contained in S). What are the tangent cones and normal cones at x? What are the necessary optimality conditions there? Does it make sense?

3. Optimizing on the unit circle. Consider the following optimization problem:

$$\min_{(x,y)\in\mathbb{R}^2} x + y \qquad \text{subject to} \qquad x^2 + y^2 = 1.$$

1. Draw the feasible set and the gradient vector field of the cost function. Based on this drawing, can you guess what the solutions are? Can you guess what the stationary points are (see lecture notes)?
2. We may be tempted to solve the problem by eliminating y with the change of variable $y = \pm\sqrt{1 - x^2}$. Do this, and show that the two possible signs lead to two different answers. Use the necessary optimality conditions (see lecture notes) to identify the right one.

4. Optimizing on the unit disk. Find the *maxima* of $f(x, y) = xy$ on the closed unit disk, defined by the inequality $x^2 + y^2 \leq 1$. To do this, consider the tangent cones to the unit disk both in the interior and on the boundary; use this to determine the stationary points. As usual, we recommend you draw the situation. In particular sketch the gradient field.

5. Tangent cone of the infinity norm ball. Consider the closed infinity norm ball of unit radius given by

$$B_\infty = \left\{ x \in \mathbb{R}^n : \|x\|_\infty = \max_{i=1,\ldots,n} |x_i| \leq 1 \right\}.$$

1. Draw this set for $n = 1$ and for $n = 2$.
2. What is the tangent cone for a point in the interior of B_∞?
3. What is the tangent cone for a point on the boundary of B_∞? Start with $n = 1$ and $n = 2$.

Supplementary exercises

1. Optimizing on the unit circle with change of variable. In an exercise above we saw that a change of variables can introduce spurious solutions. However some variable changes are better than others. In this one you will parametrize the circle using $\theta \mapsto (\cos\theta, \sin\theta)$. Consider the constrained optimization problem.

$$\min_{(x,y)\in\mathbb{R}^2} xy \quad \text{subject to} \quad x^2 + y^2 = 1$$

1. Draw the gradient of $f(x, y) = xy$ at various points of the unit circle. Based on this drawing, guess where the stationary points are, and guess which points are the minimizers.
2. Introduce the change of variable described above to obtain an optimization problem of one variable.
3. Solve this single-variable optimization problem. Explain how you use your findings to determine the minimizers of f on the circle.

2. Infinity norm minimization. Let $F\colon \mathbb{R}^n \to \mathbb{R}^m$ be a smooth vector function and consider the unconstrained optimization problem of minimizing $f(x)$ where

$$f(x) = \|F(x)\|_\infty = \max_{i=1,\ldots,m} |F_i(x)|.$$

Notice that this cost function is (generally) *not* smooth. Reformulate the problem as a *smooth* constrained optimization problem, that is: describe the search space S with smooth inequalities, and arrange for the cost function to be smooth as well. *Hint: the essential trick is to add a new, "fake" variable. You can look in the Nocedal & Wright for inspiration.*