LECTURE NOTES 'TOPICS IN COMPLEX ANALYSIS'

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0. Recapitulation of some results in complex analysis

In this section we recall some standard theorems from complex analysis that we will use in the course. We give no proofs as they can be found in (almost) any text book on complex analysis.

If not stated otherwise we shall use the following notation:

- $\mathbb{C}$: complex numbers
- U: an open subset of $\mathbb{C}$
- D: a domain (open and path-connected subset of $\mathbb{C}$)
- $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$ open ball with radius $r > 0$ and center $z_0 \in \mathbb{C}$

Definition 0.1. A function $f : U \to \mathbb{C}$ is called complex differentiable in $z_0 \in U$ if there exists the limit

$$f'(z_0) = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(z_0 + h) - f(z_0)}{h} \in \mathbb{C}.$$

It is called holomorphic on U if it is complex differentiable in every $z_0 \in U$.

Theorem 0.2 (Cauchy's integral formula)**.** *Let $f : U \to \mathbb{C}$ be holomorphic and suppose that the closed disc $\overline{B_r(z_0)}$ is contained in U. Then for every $a \in B_r(z_0)$ we have*

$$f(a) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(z)}{z - a} \, \mathrm{d}z,$$

where the circle $\partial B_r(z_0)$ is oriented counterclockwise.

Corollary 0.3 (Analyticity of holomorphic functions)**.** *Under the assumptions of Theorem 0.2 the function f is analytic on U and each $f^{(k)} : U \to \mathbb{C}$ is holomorphic with*

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(z)}{(z - a)^{k+1}} \, \mathrm{d}z.$$

Corollary 0.4 (Liouville's theorem)**.** *Every bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant.*

Theorem 0.5 (Morera's theorem)**.** *Let $f : U \to \mathbb{C}$ be continuous. If for each triangle $\Delta \subset U$ it holds that*

$$\int_{\partial \Delta} f(z) \, \mathrm{d}z = 0,$$

then f is holomorphic on U.

Theorem 0.6 (Identity theorem)**.** *Let $D \subset \mathbb{C}$ be a domain and $f, g : D \to \mathbb{C}$ be holomorphic. If the set $\{z \in \mathbb{C} : f(z) = g(z)\}$ has an accumulation point in D, then $f = g$.*

Theorem 0.7 (Open mapping theorem)**.** *Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ be a non-constant holomorphic function. Then $f(D)$ is a domain as well.*

Corollary 0.8 (Maximum principle)**.** *Let $D \subset \mathbb{C}$ be a domain and let $f : D \subset \mathbb{C}$ be a holomorphic function. If $|f|$ attains its maximum on D then f is constant.*

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Singularities of holomorphic functions. Isolated singularities of holomorphic functions are subdivided into three categories.

Definition 0.9. Let $U \subset \mathbb{C}$ be open and let $z_0 \in U$. Assume that $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic. z_0 is called

(i) a removable singularity if f can be extended to a holomorphic function $\tilde{f} : U \to \mathbb{C}$;
(ii) a pole if there exists $m \in \mathbb{N}$ such that $z \mapsto (z - z_0)^m f(z)$ has a removable singularity in z_0. The smallest such m is called the order of the pole;
(iii) an essential singularity if z_0 is neither a removable singularity nor a pole.

Theorem 0.10 (Laurent series expansion)**.** *Let $0 \leq r < R$ and let $f : \{z \in \mathbb{C} : r < |z - z_0| < R\} \to \mathbb{C}$ be holomorphic. Then f has the representation*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

where, for all $n \in \mathbb{Z}$ and $r < \rho < R$,

$$c_n = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} \,\mathrm{d}z.$$

The term $\sum_{n=-\infty}^{-1} c_n (z - z_0)^n$ is called principal part of f, while the term $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ is called regular (or holomorphic) part of f.

Corollary 0.11. *Let $f : U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Then z_0 is*

(i) a removable singularity $\iff$ $c_k = 0 \quad \forall k < 0$ $\iff$ f is bounded near z_0;
(ii) a pole of order m $\iff$ $c_k = 0 \quad \forall k < -m$ and $c_{-m} \neq 0$;
(iii) an essential singularity $\iff$ $c_k \neq 0$ for infinitely many $k < 0$.

1. Sequences of holomorphic functions

Next we consider sequences of holomorphic functions $f_n : U \to \mathbb{C}$ and their convergence properties, i.e., compactness, convergence criteria and properties of the limit. As we shall see the following notion of convergence is well-suited with regard to the above properties.

Definition 1.1. A sequence $f_n : U \to \mathbb{C}$ of holomorphic functions is said to converge locally uniformly to some function $f : U \to \mathbb{C}$ if for each $z_0 \in U$ there exists $r > 0$ such that

$$\sup_{z \in B_r(z_0)} |f_n(z) - f(z)| \to 0 \quad \text{as } n \to +\infty.$$

Remark 1.2. Local uniform convergence is equivalent to uniform convergence on each compact subset of U.

The following theorem shows that local uniform convergence preserves holomorphy.

Theorem 1.3. *Assume that a sequence $f_n : U \to \mathbb{C}$ of holomorphic functions converges locally uniformly to some $f : U \to \mathbb{C}$. Then f is holomorphic.*

Proof. Note that f is continuous as the locally uniform limit of continuous functions. Hence by Morera's theorem it suffices to check that for each triangle $\Delta \subset U$ we have

$$\int_{\partial \Delta} f(z) \,\mathrm{d}z = 0.$$

Since $f_n \to f$ uniformly on Δ by Remark 1.2 we conclude from Cauchy's theorem that

$$0 = \lim_{n \to +\infty} \int_{\partial \Delta} f_n(z) \,\mathrm{d}z = \int_{\partial \Delta} f(z) \,\mathrm{d}z,$$

where the last equality can be justified for instance by Lebesgue's dominated convergence theorem. This proves the claim. □

Remark 1.4. Theorem 1.3 is in general false for pointwise converging sequences of holomorphic functions (an example can be found in [1]). However, Osgood's theorem [4, p. 151] (see also exercise H 2.4) ensures that the pointwise limit is holomorphic on a dense, open subset of U.

For sequences $f_n : \mathbb{R} \to \mathbb{R}$ uniform convergence does not allow to conclude convergence of the derivatives. For instance, the sequence $f_n(x) = \frac{1}{n}\sin(nx)$ converges uniformly to 0, but its derivative $f_n'(x) = \cos(nx)$ does not even converge pointwise. As we prove next, holomorphic functions behave much better.

Theorem 1.5. *Let $f_n : U \to \mathbb{C}$ be a sequence of holomorphic functions that converges locally uniformly to $f : U \to \mathbb{C}$. Then for each $k \in \mathbb{N}$ the sequence $f_n^{(k)}$ converges locally uniformly to $f^{(k)}$.*

Proof. Let $z_0 \in U$ and $r > 0$ be such that $\overline{B_{2r}(z_0)} \subset U$. Due to Cauchy's integral formula, for all $z' \in B_r(z_0)$ we can write

$$f^{(k)}(z') - f_n^{(k)}(z') = \frac{k!}{2\pi i}\int_{\partial B_{2r}(z_0)} \frac{f(z) - f_n(z)}{(z-z')^{k+1}}\,\mathrm{d}z.$$

Note that for $z' \in B_r(z_0)$ and $z \in \partial B_{2r}(z_0)$ it holds that $|z - z'| \geq r$. Since the length of $\partial B_{2r}(z_0)$ equals $4\pi r$ we deduce that

$$\sup_{z' \in B_r(z_0)} |f^{(k)}(z') - f_n^{(k)}(z')| \leq \frac{2k!}{r^k} \sup_{z' \in B_{2r}(z_0)} |f(z') - f_n(z')|.$$

Due to Remark 1.2 the right hand side converges to 0 when $n \to +\infty$ and we conclude the proof. □

The previous theorem allows to control the number of zeros of the limit of holomorphic functions.

Corollary 1.6. *Let $D \subset \mathbb{C}$ be a domain and $f_n : D \to \mathbb{C}$ be a sequence of holomorphic functions that converges locally uniformly to $f : D \to \mathbb{C}$. If each f_n has at most m zeros (counted with multiplicity), then either $f \equiv 0$ or f has at most m zeros.*

Proof. Let $f \not\equiv 0$ and assume by contradiction that f has distinct zeros $z_1, \dots, z_\ell$ with total multiplicity larger than m. By the identity theorem the zeros of f are isolated, so that for each z_j we find a ball $B_r(z_j)$ such that

(i) $\{f = 0\} \cap \overline{B_r(z_j)} = \{z_j\}$,

(ii) $\overline{B_r(z_j)} \cap \overline{B_r(z_i)} = \emptyset \quad \forall 1 \leq j \neq i \leq \ell$.

The argument principle then implies

$$m + 1 \leq \sum_{j=1}^{\ell} \frac{1}{2\pi i}\int_{\partial B_r(z_j)} \frac{f'(z)}{f(z)} = \sum_{j=1}^{\ell} \lim_{n\to+\infty} \frac{1}{2\pi i}\int_{\partial B_r(z_j)} \frac{f_n'(z)}{f_n(z)}\,\mathrm{d}z \leq m,$$

where in the second equality we used Theorem 1.5 and that for n large enough we have $f_n \neq 0$ on the compact set $\partial B_r(z_j)$. This yields a contradiction. □

Next we turn our attention to convergence criteria. The first one is a general compactness result.

Theorem 1.7 (Montel's theorem)**.** *Let $f_n : U \to \mathbb{C}$ be a sequence of holomorphic functions that is locally uniformly bounded, i.e., for each $z_0 \in U$ there exists $r > 0$ and $C < +\infty$ such that*

$$\sup_{n\in\mathbb{N}} \sup_{z \in B_r(z_0)} |f_n(z)| \leq C.$$

Then there exists a subsequence f_{n_k} that converges locally uniformly to a holomorphic function $f : U \to \mathbb{C}$.

Proof. Take a countable, dense subset S of U (e.g. $(\mathbb{Q} + i\mathbb{Q}) \cap U$) and let us write $S = \{z_1, z_2, z_3, \dots\}$. Since the sequence $\{f_n(z_1)\}_{n\in\mathbb{N}}$ is bounded, we may apply the Bolzano-Weierstrass theorem in order to extract a subsequence $n_{k,1}$ such that $f_{n_{k,1}}(z_1)$ converges to some value $f_{z_1} \in \mathbb{C}$. In a next step we note that the sequence $\{f_{n_{k,1}}(z_2)\}_{k\in\mathbb{N}}$ is again bounded, so that by the same reasoning we find another subsequence $n_{k,2}$ of the previous subsequence such that $\{f_{n_{k,2}}(z_2)\}_{k\in\mathbb{N}}$ converges to some value $f_{z_2} \in \mathbb{C}$.

In the j^{th} step we choose a subsequence $n_{k,j}$ of all previous subsequences such that $\{f_{n_{k,j}}\}_{k\in\mathbb{N}}$ converges to some value $f_{z_j}\in\mathbb{C}$. For $k\in\mathbb{N}$ we finally set $n_k:=n_{k,k}$. Then the sequence $f_{n_k}(z_j)$ converges to f_{z_j} for all $j\in\mathbb{N}$ since except for finitely many terms the sequence n_k is a subsequence of $\{n_{k,j}\}_{k\in\mathbb{N}}$. Thus we found a subsequence f_{n_k} such that $f_{n_k}(z)$ converges to some value $f_z\in\mathbb{C}$.

Next we show that f_n is equicontinuous, i.e., the $\varepsilon-\delta$ definition of continuity is valid with δ independent of n. Fix $z_0\in U$ and let $r>0$ be such that $\overline{B_{2r}(z_0)}\subset U$ and such that there exists $C<+\infty$ with

$$\sup_{n\in\mathbb{N}}\sup_{z\in\overline{B_{2r}(z_0)}}|f_n(z)|\leq C.$$

By Cauchy's integral formula, for all $z'\in B_r(z_0)$ we have

$$\begin{aligned}|f_n(z')-f_n(z_0)|=&\left|\frac{1}{2\pi i}\int_{\partial B_{2r}(z_0)}\frac{f_n(z)}{z-z'}-\frac{f_n(z)}{z-z_0}\,\mathrm{d}z\right|=\frac{|z'-z_0|}{2\pi}\left|\int_{\partial B_{2r}(z_0)}\frac{f_n(z)}{(z-z')(z-z_0)}\,\mathrm{d}z\right|\\ \leq&\frac{|z'-z_0|}{2\pi}\frac{C\cdot 4\pi r}{2r^2}=\frac{C|z'-z_0|}{r},\end{aligned}$$

where we used that $|z-z'|\geq r$ and $|z-z_0|\geq 2r$ for all $z\in\partial B_{2r}(z_0)$. The right hand side is independent of n, so given $\varepsilon>0$ we can choose $\delta_\varepsilon=\min\{r,\varepsilon\frac{r}{C}\}$ in the definition of continuity. Hence f_n is equicontinuous.

Equicontinuity allows us to show that $\{f_{n_k}(z)\}_{k\in\mathbb{N}}$ is a Cauchy sequence for all $z\in U$. To reduce notation, we skip the subscript k. For $z\in U$ and $\varepsilon>0$ we first choose $z^*\in S$ such that $|z-z^*|<\delta_{\varepsilon,z}$, where $\delta_{\varepsilon,z}$ satisfies the equicontinuity condition

$$|y-z|<\delta_{\varepsilon,z}\quad\Rightarrow\quad|f_n(y)-f_n(z)|<\frac{\varepsilon}{3}\qquad\forall n\in\mathbb{N}.\tag{1}$$

To find such a z^* is possible due to the density of S in U. For $m\geq n$ we then have

$$|f_m(z)-f_n(z)|\leq\underbrace{|f_m(z)-f_m(z^*)|}_{<\varepsilon/3}+|f_m(z^*)-f_n(z^*)|+\underbrace{|f_n(z^*)-f_n(z)|}_{<\varepsilon/3}$$

Since $z^*\in S$ the convergence on S implies that there exists $n_\varepsilon\in\mathbb{N}$ such that for all $m\geq n\geq n_\varepsilon$ we have $|f_m(z^*)-f_n(z^*)|<\frac{\varepsilon}{3}$. Then for all $m\geq n\geq n_\varepsilon$ we conclude that

$$|f_m(z)-f_n(z)|<\varepsilon.$$

Hence $\{f_n(z)\}_{n\in\mathbb{N}}$ is Cauchy sequence as claimed, so that there exists $f_z=\lim_{n\to+\infty}f_n(z)$ for all $z\in U$.

Finally, we show that f_n converges locally uniformly to $f(z):=f_z$. Fix a compact set $K\subset U$. Given $\varepsilon>0$ and $z\in K$ we choose $\delta_{\varepsilon,z}>0$ satisfying (1) above. Then the family of discs $\{B_{\delta_{\varepsilon,z}}(z)\}_{z\in K}$ forms an open cover of K. By the (topological) definition of compactness there exists a finite sub-family $\{B_{\delta_{\varepsilon,z_i}}\}_{i=1}^N$ with $z_i\in K$ that still covers K. Thus for any $z\in K$ we find z_i such that $|z-z_i|<\delta_{\varepsilon,z_i}$. Since the $\{z_i\}$ are only finitely many there exists $n_\varepsilon\in\mathbb{N}$ such that for all $n\geq n_\varepsilon$ it holds that

$$|f(z_i)-f_n(z_i)|<\frac{\varepsilon}{3}.$$

Moreover, observe that (1) also holds for the limit function f as we can pass to the limit in this estimate. Consequently, for $n\geq n_\varepsilon$ we deduce that for all $z\in K$ we have

$$|f(z)-f_n(z)|\leq\underbrace{|f(z)-f(z_i)|}_{\varepsilon/3}+\underbrace{|f(z_i)-f_n(z_i)|}_{<\varepsilon/3}+\underbrace{|f_n(z_i)-f_n(z)|}_{<\varepsilon/3}<\varepsilon,$$

which shows the uniform convergence of f_n to f and we conclude the proof. □

Finally, we state two criteria which ensure the convergence along the whole sequence.

Theorem 1.8 (Vitali's theorem)**.** *Let $D\subset\mathbb{C}$ be a domain and let $f_n:D\to\mathbb{C}$ be a sequence of holomorphic functions that is locally uniformly bounded. If the set $L:=\{z\in D:\lim_{n\to+\infty}f_n(z)$ exists$\}$ has an accumulation point in D, then f_n converges locally uniformly to some holomorphic function $f:D\to\mathbb{C}$.*

Proof. Due to Montel's theorem there exists a subsequence f_{n_k} that converges locally uniformly to a holomorphic function $f : D \to \mathbb{C}$. Note that local uniform convergence is induced by a topology. Hence the non-convergence of the whole sequence to f implies that there exists another subsequence which has no subsequence that does converge locally uniformly to f. Applying Montel's theorem along this subsequence, we obtain another subsequence $f_{n_{k,1}}$ that converges locally uniformly to a holomorphic function $h : D \to \mathbb{C}$. Then $h \neq f$, but $h(z) = f(z)$ for all $z \in L$, which contradicts the identity theorem. □

Theorem 1.9. *Let $D \subset \mathbb{C}$ be a domain and let $f_n : D \to \mathbb{C}$ be a sequence of holomorphic functions that is locally uniformly bounded. If for all $k \in \mathbb{N} \cup \{0\}$ and some $z_0 \in D$ the sequences $f_n^{(k)}(z_0)$ converge, then f_n converges locally uniformly to some holomorphic function $f : D \to \mathbb{C}$.*

Proof. See Exercise H 2.1. □

Local normal convergence. In the next chapter the focus will be on series of holomorphic functions. For those the following concept of convergence will be useful.

Definition 1.10. Let $f_j : U \to \mathbb{C}$ be a sequence of complex-valued functions. The series $\sum_{j=1}^{\infty} f_j$ is called locally normally convergent if for each $z_0 \in U$ there exists $r > 0$ such that

$$\sum_{j=1}^{\infty} \sup_{z \in B_r(z_0)} |f_j(z)| < +\infty.$$

As shown in the lemma below, local normal convergence implies local uniform convergence. In the exercises we will see that the converse is false in general.

Lemma 1.11. *Let $f_j : U \to \mathbb{C}$ be a sequence of complex-valued functions. If the series $\sum_{j=1}^{\infty} f_j$ converges locally normally, then it also converges locally uniformly. In particular, if each f_j is in addition holomorphic, then $z \mapsto \sum_{j=1}^{\infty} f_j(z)$ is holomorphic, too.*

Proof. See exercise H 3.4 and Theorem 1.3 □

2. The Mittag-Leffler theorem

We start with the following simple observation: If $\{d_1, \dots, d_n\} \subset \mathbb{C}$ is a finite set and for each d_n the function $q_n : \mathbb{C} \setminus \{d_n\} \to \mathbb{C}$ denotes a finite principle part at d_n given by $q_n(z) = \sum_{j=1}^{m_n} a_{nj}(z - d_n)^{-j}$, then the function

$$f(z) = \sum_{n=1}^{N} q_n(z)$$

is meromorphic on $\mathbb{C}$ and at each $d_n \in \mathbb{C}$ the principal part of its Laurent series agrees with q_n. In 1876/77 the Swedish mathematician Gösta Mittag-Leffler extended the above result to (in $\mathbb{C}$) discrete sets $\{d_n\}_{n \in \mathbb{N}}$ (i.e. with no accumulation point). In 1880 Karl Weierstraß found a simplified proof which in general also allows for $m_n = +\infty$ (albeit some implicit growth conditions on the coefficients a_{nj} by requiring that $q_n : \mathbb{C} \setminus \{d_n\}$ is holomorphic). In this course we shall follow the argument of Weierstrass but prove a more general version valid on open sets. To reduce notation, we introduce some vocabulary.

Definition 2.1. Let $d \in \mathbb{C}$ and $q : \mathbb{C} \setminus \{d\} \to \mathbb{C}$ be a holomorphic function. q is called a principal part at d when its Laurent series expansion around d has no regular part.

With this definition the theorem of Mittag-Leffler on $\mathbb{C}$ reads as follows:

Theorem 2.2 (Mittag-Leffler on $\mathbb{C}$). *Let $S = \{d_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$ be a discrete set. For each $d_n \in S$ let $q_n : \mathbb{C} \setminus \{d_n\} \to \mathbb{C}$ be a principal part. Then there exists a holomorphic function $f : \mathbb{C} \setminus S \to \mathbb{C}$ such that at each $d_n \in S$ its principal part of the Laurent series is given by q_n. The function f can be taken to be of the form*

$$f(z) = \sum_{n=1}^{\infty} q_n(z) - p_n(z),$$

where $p_n : \mathbb{C} \to \mathbb{C}$ is a polynomial and the sum converges locally normally on $\mathbb{C} \setminus S$.

Proof. Without loss of generality we may assume that $0 < |d_1| \leq |d_2| \leq \dots$ (if $d_1 = 0$ then set $p_1 \equiv 0$ and separate this term from the analysis). Once cannot expect that the series $\sum_{n=1}^{\infty} q_n$ converges. Hence we have to use the polynomials to ensure convergence. As they should be close to q_n we take suitable Taylor-polynomials. Since q_n is holomorphic on $B_{|d_n|}(0)$ it has a convergent series expansion on this ball, i.e., for any $z \in B_{|d_n|}(0)$ it holds that

$$q_n(z) = \sum_{j=0}^{\infty} a_{nj}\, z^j .$$

By general properties of power series we know that on the smaller ball $B_{1/2|d_n|}(0)$ the above series converges uniformly. Hence for each $n \in \mathbb{N}$ there exists a number $j_n \in \mathbb{N}$ such that

$$\sup_{z \in B_{1/2|d_n|}(0)} \Big| q_n(z) - \underbrace{\sum_{j=0}^{j_n} a_{nj}\, z^j}_{=:p_n(z)} \Big| \leq 2^{-n}. \tag{2}$$

Next fix a compact set $K \subset \mathbb{C} \setminus S$. Due to the discreteness of S we know that $\lim_n |d_n| = +\infty$. As K is in particular bounded we find a number $n(K) \in \mathbb{N}$ such that for all $n \geq n(K)$ we have $K \subset B_{1/2|d_n|}(0)$. Consequently, from (2) we infer that

$$\sum_{n \geq n(K)} \sup_{z \in K} |q_n(z) - p_n(z)| \leq \sum_{n \geq n(K)} \sup_{z \in B_{1/2|d_n|}(0)} |q_n(z) - p_n(z)| \leq \sum_{n \geq n(K)} 2^{-n} < +\infty.$$

Since K is a compact subset of $\mathbb{C} \setminus S$ all functions $q_n - p_n$ are bounded on K. Hence we have shown that the series

$$f := \sum_{n=1}^{\infty} q_n - p_n$$

converges locally normally on $\mathbb{C} \setminus S$. In particular, by Lemma 1.11 it is holomorphic. Finally, in order to obtain the principal part at a point d_n we argue as follows: choose $\rho > 0$ such that $B_{2\rho}(d_n) \cap S = \{d_n\}$. Then by the formula for the Laurent coefficients and local uniform convergence of f we deduce that the jth Laurent coefficient at d_n, denoted here by $a_j(d_n)$, is given by

$$a_j(d_n) = \frac{1}{2\pi i} \int_{\partial B_\rho(d_n)} \frac{f(z)}{(z - d_n)^{j+1}}\, \mathrm{d}z = \sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_{\partial B_\rho(d_n)} \frac{q_k(z) - p_k(z)}{(z - d_n)^{j+1}}\, \mathrm{d}z.$$

For $j \leq -1$ the only integrand that is not holomorphic on $B_{2\rho}(d_n)$ is $q_n(z)(z - d_n)^{-(j+1)}$. All other contributions vanish due to Cauchy's integral theorem. Hence for $j \leq -1$ we deduce that

$$a_j(d_n) = \frac{1}{2\pi i} \int_{\partial B_\rho(d_n)} \frac{q_n(z)}{(z - d_n)^{j+1}}\, \mathrm{d}z,$$

which coincides with the jth Laurent coefficient at d_n of q_n. Thus the principal part at d_n is given by q_n as claimed. □

Remark 2.3. Any other holomorphic function $\tilde{f} : \mathbb{C} \setminus S \to \mathbb{C}$ with the same principal parts at $d_n \in S$ differs from f by an entire function. Indeed, the difference $f - \tilde{f}$ has removable singularities at each $d_n \in S$ since all its Laurent coefficients with negative index vanish.

Now we prepare for extending the previous theorem to general open sets $U \subset \mathbb{C}$ and point sets $S = \{d_n\}_{n \in \mathbb{N}} \subset U$, which are discrete in U (but may have accumulation points at ∂U). Instead of polynomials we will use truncated Laurent series to ensure convergence of the series.

Definition 2.4. Given a principal part $q(z) = \sum_{j=1}^{\infty} a_{-j}(z - d)^{-j}$ and $k \in \mathbb{N}$ we define the truncated Laurent series $q^k(z) = \sum_{j=1}^{k} a_{-j}(z - d)^{-j}$.

Before we prove the general Mittag-Leffler theorem for special sets S we need some auxiliary results concerning principal parts.

Lemma 2.5. *Let $q : \mathbb{C} \setminus \{d\} \to \mathbb{C}$ be holomorphic.*

(i) q is a principal part $\iff$ $\lim_{|z|\to+\infty} q(z) = 0$.

(ii) If q is a principal part then on the annulus $A_c = \{z \in \mathbb{C} : |z-c| > |d-c|\}$, $c \in \mathbb{C}$, the function q admits a Laurent series expansion of the form

$$q(z) = \sum_{j=1}^{\infty} \tilde{a}_{-j}(z-c)^{-j}.$$

Proof. (i) By Theorem 0.10 we can write $q = q^+ + q^-$, where the regular part $q^+ : \mathbb{C} \to \mathbb{C}$ is an entire function and the principal part $q^- : \mathbb{C} \setminus \{d\} \to \mathbb{C}$ is of the form $q^-(z) = \sum_{j=1}^{\infty} a_{-j}(z-d)^{-j}$. First we recall a general bound on the coefficients of a Laurent series expansion. In our case, for all $\rho > 0$ and $j \in \mathbb{Z}$ it holds that

$$|a_j| = \left| \frac{1}{2\pi i} \int_{\partial B_\rho(d)} \frac{q(z)}{(z-d)^{j+1}} \right| \leq \frac{1}{2\pi} \underbrace{\operatorname{Length}(\partial B_\rho(d))}_{=2\pi\rho} \frac{\sup_{z\in\partial B_\rho(d)} |q(z)|}{\rho^{j+1}} \leq \sup_{z\in\partial B_\rho(d)} |q(z)|\, \rho^{-j}. \tag{3}$$

Inserting $\rho = 1$ we see that $|a_j| \leq C$ for some constant independent of j. From that it is not difficult to prove that

$$\lim_{|z|\to+\infty} q^-(z) = 0. \tag{4}$$

If q is a principal part, then $q^+ \equiv 0$ so that (4) implies that $\lim_{|z|\to+\infty} q(z) = 0$. On the other hand, if $\lim_{|z|\to+\infty} q(z) = 0$ then (4) yields that $\lim_{|z|\to+\infty} q^+(z) = 0$. Since q^+ is an entire function we deduce from Liouville's theorem that $q^+ \equiv 0$, so that q is a principal part. This proves the equivalence (i).

(ii) In order to prove the second statement, note that since q is holomorphic on $A_c \subset \mathbb{C} \setminus \{d\}$ it admits a Laurent series representation centered at c. Similar to (3) the coefficients satisfy for any $\rho > |d-c|$ the estimate

$$|\tilde{a}_j| \leq \sup_{z\in\partial B_\rho(c)} |q(z)|\, \rho^{-j}$$

For $\rho \gg 1$ the factor ρ^{-j} is bounded for all $j \geq 0$. By (i) we know that the supremum vanishes when $\rho \to +\infty$. Hence $\tilde{a}_j = 0$ for all $j \geq 0$ as claimed. □

Now we can prove the general Mittag-Leffler theorem for a special class of sets S. First some notation. Given a set $S \subset U$ that is discrete in U, we define $S' = \overline{S} \setminus S$ as the set of its accumulation points in $\mathbb{C}$.

Proposition 2.6. *Let $S = \{d_n\}_{n\in\mathbb{N}} \subset U$ be a countable set that is discrete in U and for each $d_n \in S$ let $q_n : \mathbb{C} \setminus \{d_n\} \to \mathbb{C}$ be a principal part. If there exists a sequence $\{c_n\}_{n\in\mathbb{N}} \subset S'$ such that $\lim_n |d_n - c_n| = 0$ then there exist truncated principal parts $q_n^{k_n}$ centered in c_n (cf. Lemma 2.5(ii)) such that the series*

$$f = \sum_{n=1}^{\infty} q_n - q_n^{k_n}$$

converges locally normally in $\mathbb{C} \setminus \overline{S} \supset U \setminus S$ and at each point $d_n \in S$ the principal part of f is given by q_n.

Proof. Note that by Lemma 2.5(ii) and general properties of Laurent series the sequence of truncated Laurent series centered at c_n converges uniformly to q_n on the smaller annulus $A^2_{c_n} := \{z \in \mathbb{C} : |z-c_n| \geq 2|d_n - c_n|\}$. Hence for each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that

$$|q_n(z) - q_n^{k_n}(z)| \leq 2^{-n} \quad \forall z \in A^2_{c_n}. \tag{5}$$

Now fix a compact subset K of $\mathbb{C} \setminus \overline{S}$. Then $\operatorname{dist}(\overline{S}, K) > \varepsilon$ for some $\varepsilon > 0$ and since $\lim_n |d_n - c_n| = 0$ we find an index $n(K) \in \mathbb{N}$ such that for all $n \geq n(K)$ it holds that

$$K \subset A^2_{c_n} = \{z \in \mathbb{C} : \underbrace{|z-c_n|}_{\geq\varepsilon \text{ on } K} \geq \underbrace{2|d_n - c_n|}_{\to 0}\},$$

where we recall that $c_n \in S' \subset \overline{S}$. Consequently we can use (5) to estimate

$$\sum_{n \geq n(K)} \sup_{z \in K} |q_n(z) - q_n^{k_n}(z)| \leq \sum_{n \geq n(K)} 2^{-n} < +\infty.$$

Since the singularities of q_n and $q_n^{k_n}$ are contained in $\overline{S}$ all functions $q_n - q_n^{k_n}$ are bounded on K. Hence we have shown the local normal convergence of

$$f = \sum_{n=1}^{\infty} q_n - q_n^{k_n}.$$

In particular, f is holomorphic on $\mathbb{C} \setminus \overline{S}$. Finally, since each $q_n^{k_n}$ is holomorphic on U (recall that we have chosen the centers of the truncated Laurent series in $S' \subset \partial U$), by the same reasoning as for Theorem 2.2 we conclude that at each $d_n \in S$ the principal part of f is given by q_n. $\square$

As a next step we divide the set of singularities S in suitable way so that for one subset we can apply the Mittag-Leffler theorem on $\mathbb{C}$ and on the other set we apply the special version above. The basic idea is to split the points into a closed set and sets close to an accumulation point. The following lemma makes this splitting precise.

Lemma 2.7. *Let $S \subset U$ be a discrete set in U such that $S' = \overline{S} \setminus S \neq \emptyset$. Define*

$$S_1 := \{z \in S : \, |z|\mathrm{dist}(S', z) \geq 1\}, \qquad S_2 := \{z \in S : \, |z|\mathrm{dist}(S', z) < 1\}.$$

Then S_1 is closed and for every $\varepsilon > 0$ the set $S_2(\varepsilon) := \{z \in S_2 : \, \mathrm{dist}(S', z) \geq \varepsilon\}$ is finite.

Proof. We first prove that S_1 is closed. Let $\{z_n\}_{n \in \mathbb{N}} \subset S_1$ be a sequence such that $z_n \to z^*$ for some $z^* \in \mathbb{C}$. Due to the continuity of the function $z \mapsto |z|\mathrm{dist}(S', z)$ we know that $|z^*|\mathrm{dist}(S', z^*) \geq 1$. We claim that $z^* \in S$ which shows that S_1 is closed. Indeed, assume by contradictions that $z^* \notin S$. Then by definition $z^* \in S'$, which contradicts the fact that $|z^*|\mathrm{dist}(S', z^*) \geq 1$.

In order to prove the second assertion, note that for any $z \in S_2(\varepsilon)$ we have by definition

$$|z| \leq \mathrm{dist}(S', z)^{-1} \leq \frac{1}{\varepsilon}.$$

Thus, assuming by contradiction that the cardinality of $S_2(\varepsilon)$ is infinite for some $\varepsilon > 0$, there exists a sequence of distinct points $\{z_n\}_{n \in \mathbb{N}} \subset S_2(\varepsilon)$ such that $z_n \to z^*$ for some $z^* \in \overline{S}$. Since $S \subset U$ does not contain any accumulation point it follows that $z^* \in S'$. But due to continuity it holds that $\mathrm{dist}(S', z^*) \geq \varepsilon$, which yields a contradiction. $\square$

The splitting $S = S_1 \cup S_2$ can be further justified by the following property which will allow us to apply Proposition 2.6.

Lemma 2.8. *Let $S_2 = \{d_n\}_{n \in \mathbb{N}} \subset U$ be as in Lemma 2.7 and assume that $S_2' \neq \emptyset$. Then there exists a sequence $\{c_n\}_{n \in \mathbb{N}} \subset S_2'$ such that $\lim_n |d_n - c_n| = 0$.*

Proof. First note that S_2' is closed (this is a general fact which can be proven by a diagonal argument). Hence for each $n \in \mathbb{N}$ there exists $c_n \in S_2'$ such that $\mathrm{dist}(S_2', d_n) = |d_n - c_n|$. If the latter term does not converge to zero, then for some $\varepsilon > 0$ the cardinality of $S_2(\varepsilon)$ defined in Lemma 2.7 would be infinite. Indeed, the assumption $S_2' \neq \emptyset$ implies that the cardinality of S_2 is infinite. Moreover, we have that $S' = S_2'$ since the set S_1 is closed. $\square$

Now we can state and prove the full theorem of Mittag-Leffler on open sets, which will be the final result of this chapter.

Theorem 2.9. *Let $U \subset \mathbb{C}$ be open and let $S = \{d_n\}_{n \in \mathbb{N}} \subset U$ be discrete in U. For each d_n let $q_n : \mathbb{C} \setminus \{d_n\} \to \mathbb{C}$ be a principal part. Then there exists a holomorphic function $f : U \setminus S \to \mathbb{C}$ such that at each d_n its principal part is given by q_n. The function f can be taken to be of the form*

$$f = \sum_{n=1}^{\infty} q_n - h_n,$$

where each $h_n : U \to \mathbb{C}$ is holomorphic and the series converges locally normally on $U \setminus S$.

Proof. Without loss of generality we may assume that $S' = \overline{S} \setminus S \neq \emptyset$ (otherwise S is discrete in $\mathbb{C}$ and we can apply Theorem 2.2). Let S_1 and S_2 be defined as in Lemma 2.7. Since S_1 is closed by that lemma we know that $S_1' = \emptyset$ and therefore $S_2' = S'$. Let us write $S_1 = \{d_{n,1}\}_n$ and $S_2 = \{d_{n,2}\}_n$ (we don't claim that both are infinite). Since $S_1 \subset S$ is discrete in $\mathbb{C}$ we can apply Theorem 2.2 to deduce that there exists a family of polynomials $p_{n,1} : \mathbb{C} \to \mathbb{C}$ such that

$$f_1 = \sum_{d_{n,1} \in S_1} q_{n,1} - p_{n,1}$$

is holomorphic on $\mathbb{C} \setminus S_1$, at each $d_{n,1} \in S_1$ its principle part is given by $q_{n,1}$ and the series converges locally normally on $\mathbb{C} \setminus S_1$.

Next we treat the set S_2. On this set we can apply Proposition 2.6 (cf. Lemmata 2.7 & 2.8) to deduce that there exists holomorphic functions $h_{n,2} : U \to \mathbb{C}$ such that

$$f_2 = \sum_{d_{n,2} \in S_2} q_{n,2} - h_{n,2}$$

is holomorphic on $U \setminus S_2$, at each $d_{n,2} \in S_2$ its principle part is given by $q_{n,2}$ and the series converges locally normally on $U \setminus S_2$.

Since $S = S_1 \dot\cup S_2$ the function $f = f_1 + f_2$ thus satisfies all the claimed properties. □

3. Infinite products

In this chapter we deal with the counterpart of series for products. The definition of infinite products $\prod_{j \geq 1} a_j$ seems quite obvious considering the Cauchy-criterion for finite partial products. However, it is customary to exclude some cases, for instance when some factors a_j equal zero or also when the limit equals zero. In this course we allow for the first case. Then the definition reads as follows:

Definition 3.1. Let $\{a_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ be a sequence of complex numbers. The infinite product $\prod_{j=1}^{\infty} a_j$ is said to converge if there exists $j_0 \in \mathbb{N}$ such that $a_j \neq 0$ for all $j \geq j_0$ and there exists the limit

$$a(j_0) := \lim_{m \to +\infty} \prod_{j=j_0}^{m} a_j \neq 0.$$

In this case we set $\prod_{j=1}^{\infty} a_j = a(j_0) \prod_{j=1}^{j_0 - 1} a_j$. Note that this definition is independent of the number j_0.

With the above definition an infinite product is zero if and only if one factor is zero. Moreover, similar to series we have a simple necessary condition for convergence.

Lemma 3.2. *Assume that the infinite product $a = \prod_{j=1}^{\infty} a_j$ converges. Then for all $m \in \mathbb{N}$ the infinite product $a(m) = \prod_{j=m}^{\infty} a_j$ exists. Moreover, $\lim_{m \to +\infty} a(m) = 1$ and $\lim_{j \to +\infty} a_j = 1$.*

Proof. Without loss of generality we may assume that $a_j \neq 0$ for all $j \in \mathbb{N}$. The existence of the products $a(m)$ follows from the definition. Moreover we have

$$\frac{a}{a(m)} = \lim_{n \to +\infty} \frac{\prod_{j=1}^{n} a_j}{\prod_{j=m}^{n} a_j} = \underbrace{\prod_{j=1}^{m-1} a_j}_{\to a \text{ as } m \to +\infty}.$$

Letting $m \to +\infty$, we deduce that

$$\lim_{m \to +\infty} a(m) a = a.$$

Since $a \neq 0$ by definition we deduce that $\lim_{m \to +\infty} a(m) = 1$. Moreover, since $a_j = a(j)/a(j+1)$ we also conclude that $\lim_{j \to +\infty} a_j = 1$ □

Next we prove an elementary, but useful criterion for the convergence of infinite products. Without loss of generality we shall assume that all factors differ from the non-positive real axis.

Lemma 3.3. *Let $\{a_j\}_{j\in\mathbb{N}}\subset\mathbb{C}\setminus\{(-\infty,0]\}$ be a sequence. Then $\prod_{j=1}^{\infty}a_j$ exists if and only if the series $\sum_{j=1}^{\infty}\log(a_j)$ exists, where* log *denotes the principal branch of the logarithm.*

Proof. First assume that $\sum_{j=1}^{\infty}\log(a_j)$ exists. Then taking the complex exponential we deduce that

$$0\neq\exp\Big(\sum_{j=1}^{\infty}\log(a_j)\Big)=\lim_{n\to+\infty}\exp\Big(\sum_{j=1}^{n}\log(a_j)\Big)=\lim_{n\to+\infty}\prod_{j=1}^{n}a_j.$$

This proves the convergence of the infinite product since we assumed that all factors are different from zero.

To prove the reverse direction, we set $P_n=\prod_{j=1}^{n}a_j$. It seems natural to take the logarithm of P_n. However, the equality $\log(z_1z_2)=\log(z_1)+\log(z_2)$ is only valid up to an additive multiple of 2π on $\mathbb{C}\setminus\{0\}$. Nevertheless, since $\prod_{j=1}^{\infty}a_j\neq0$ we can find $n_0\in\mathbb{N}$ such that for all $m\geq n\geq n_0$ it holds that

$$|P_n-P_m|\leq\frac{1}{2}\left|P_n\right|,$$

or equivalently,

$$\left|1-\frac{P_m}{P_n}\right|\leq\frac{1}{2}.$$

In particular, all the products $\prod_{j=n+1}^{m}a_j$ are contained in the right half-plane. Hence we have by the definition of the principal branch of the logarithm that

$$\log\Big(\prod_{j=n_0+1}^{m}a_j\Big)=\sum_{j=n_0+1}^{m}\log(a_j).$$

Passing to the limit as $m\to+\infty$ we deduce the claim from the continuity of $z\mapsto\log(z)$ on the right half-plane. □

For infinite products defining absolute convergence as the convergence of $\prod_{j=1}^{\infty}|a_j|$ is not beneficial. On the one hand, it would not imply the convergence of $\prod_{j=1}^{\infty}a_j$ (for instance, take $a_j=(-1)^j$). On the other hand, the convergence of $\prod_{j=1}^{\infty}a_j$ always implies the convergence of $\prod_{j=1}^{\infty}|a_j|$ due to the property $|a\cdot b|=|a|\cdot|b|$. However, Lemma 3.3 motivates the following definition.

Definition 3.4. An infinite product $\prod_{j=1}^{\infty}a_j$ is called absolutely convergent when there exists an $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ we have $a_n\notin(-\infty,0]$ and if the series $\sum_{j=n_0}^{\infty}\log(a_j)$ is absolutely convergent.

With this definition absolute convergence implies convergence by the corresponding result for series and Lemma 3.3. Moreover, we can formulate the second useful convergence criterion.

Lemma 3.5. *An infinite product $\displaystyle\prod_{j=1}^{\infty}(1+a_j)$ converges absolutely if and only if $\displaystyle\sum_{j=1}^{\infty}|a_j|$ converges.*

Proof. See Exercise H 4.2. □

Next we deal with infinite products of (holomorphic) functions. Given a sequence $f_j:U\to\mathbb{C}$ we distinguish two types of convergence of the product $\prod_{j=1}^{\infty}f_j$: local uniform and local normal convergence (cf. the corresponding notions for series).

Definition 3.6. Let $f_j:U\to\mathbb{C}$ be a sequence of continuous functions. An infinite product $\prod_{j=1}^{\infty}f_j:U\to\mathbb{C}$ is called locally uniformly convergent if for every $z_0\in U$ there exist $r>0$ and $j_0\in\mathbb{N}$ such that $\prod_{j=j_0}^{n}f_j$ converges uniformly on $B_r(z_0)$ to some non-vanishing function.

It follows from the definition that a locally uniformly convergent product converges also pointwise. There are further immediate consequences that we summarize in the corollary below.

Corollary 3.7. *Let $f_j : U \to \mathbb{C}$ be a sequence of continuous functions. Assume that $f = \prod_{j=1}^{\infty} f_j : U \to \mathbb{C}$ converges locally uniformly.*

(i) Then the sequence $\prod_{j=n}^{\infty} f_j$ converges locally uniformly to 1 as $n \to +\infty$. In particular, we have that $f_j \to 1$ locally uniformly as $j \to +\infty$;
(ii) if $\prod_{j=1}^{\infty} g_j : U \to \mathbb{C}$ is also locally uniformly converging, then so is $\prod_{j=1}^{\infty} f_j g_j$;
(iii) if each f_j is holomorphic then so is $\prod_{j=1}^{\infty} f_j$.

Proof. (i) By Lemma 3.2 the sequence $g_n(z) = \prod_{j=n}^{\infty} f_j(z)$ is pointwise well-defined. Fix $z_0 \in U$ and let $r > 0$ and $j_0 \in \mathbb{N}$ be as in Definition 3.6. The continuity of each f_j and the local uniform convergence imply that g_{j_0} is continuous. Since $g_{j_0}(z) \neq 0$ for all $z \in B_r(z_0)$ it holds that

$$\inf_{z \in B_{r/2}(z_0)} |g_{j_0}(z)| =: 2c > 0.$$

Since $\prod_{j=j_0}^{n-1} f_j(z) \to g_{j_0}$ uniformly on $B_r(z_0)$ we find an index $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\inf_{z \in B_{r/2}(z_0)} \left| \prod_{j=j_0}^{n-1} f_j(z) \right| > c.$$

Then, for $n > \max\{j_0, n_0\}$ and $z \in B_{r/2}(z_0)$,

$$|g_n(z) - 1| \leq \left| \frac{g_{j_0}(z)}{\prod_{j=j_0}^{n-1} f_j(z)} - 1 \right| \leq \frac{1}{c} \left| g_{j_0}(z) - \prod_{j=j_0}^{n-1} f_j(z) \right|.$$

This proves the first part of (i) as by assumption the last term vanishes uniformly on $B_{r/2}(z_0)$ for $n \to +\infty$. The second one follows from the first one since $f_j = \frac{g_j}{g_{j+1}}$.

(ii) This follows essentially from the definition since the product of two locally uniformly converging sequences that are locally equibounded still converges locally uniformly.

(iii) Apply Theorem 1.3. □

One drawback of local uniform convergence of products is that there is no invariance under rearrangement, i.e., the limit of infinite products might depend on the order of the sequence f_j. Hence we introduce the more stable notion of local normal convergence relying on Lemma 3.5.

Definition 3.8. An infinite product of the form $\prod_{j=1}^{\infty}(1 + g_j)$ with $g_j : U \to \mathbb{C}$ is called locally normally convergent if the series $\sum_{j=1}^{\infty} g_j$ is locally normally convergent.

We next prove that local normal convergence of products implies local uniform convergence.

Lemma 3.9. *Assume that the product $\prod_{j=1}^{\infty}(1 + g_j)$ converges locally normally. Then it converges also locally uniformly.*

Proof. Fix $z_0 \in U$ and let $r > 0$ be such that

$$\sum_{j=1}^{\infty} \sup_{z \in B_r(z_0)} |g_j(z)| < +\infty.$$

Then there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ we have $\sup_{z \in B_r(z_0)} |g_j(z)| < \delta$, where $0 < \delta < \frac{1}{2}$ is chosen such that

$$\frac{1}{2}|z| \leq |\log(1 + z)| \leq 2|z| \qquad \forall z \in B_\delta(0).$$

In particular, we have

$$\sum_{j \geq j_0} \sup_{z \in B_r(z_0)} |\log(1 + g_j(z)| < +\infty.$$

Hence by Lemma 1.11 the series $\sum_{j \geq j_0} \log(1+g_j)$ converges uniformly on $B_r(z_0)$. Taking the exponential yields the claim as the exponential function never vanishes and is continuous. □

Remark 3.10. If $\prod_{j=1}^{\infty}(1+g_j)$ converges locally normally, then taking the logarithm of $\prod_{j=j_0}^{\infty}(1+g_j)$ for a suitable large j_0 we also see that local normal convergence is invariant under rearrangements of the sequence $\{g_j\}_{j\in\mathbb{N}}$.

In Chapter 4 we will analyze the zeros of infinite products that converge locally normally. Given a holomorphic function $f: U \to \mathbb{C}$ we denote by $Z(f)$ the set of its zeros and by $o_c(f) \in \mathbb{N} \cup \{0, +\infty\}$ the order of a zero $c \in U$ (with the convention that $o_c(f) = 0$ means $f(c) \neq 0$, while $o_c(f) = +\infty$ if and only if f vanishes in a neighborbood of c).

Note that if $f_1, \dots, f_N : U \to \mathbb{C}$ is a finite family of such functions, then

$$Z(f_1 \cdot \ldots \cdot f_N) = \bigcup_{i=1}^{N} Z(f_j), \qquad o_c(f_1 \cdot \ldots \cdot f_N) = \sum_{i=1}^{N} o_c(f_i).$$

In the proposition below we generalize this result to infinite products that converge locally uniformly.

Lemma 3.11. *Let $f_j : U \to \mathbb{C}$ be a sequence of holomorphic functions. Assume that $f = \prod_{j=1}^{\infty} f_j$ converges locally uniformly. Then*

$$Z(f) = \bigcup_{j=1}^{\infty} Z(f_j), \qquad o_c(f) = \sum_{j=1}^{\infty} o_c(f_j) \quad \forall c \in U.$$

Proof. Fix $c \in U$. Since $\prod_{j=1}^{\infty} f_j(c)$ converges there exists $j_0 \in \mathbb{N}$ such that $f_j(c) \neq 0$ for all $j \geq j_0$. Write

$$f = f_1 \cdot \ldots \cdot f_{j_0-1} \cdot \underbrace{\prod_{j=j_0}^{\infty} f_j}_{=:g}$$

Since g is holomorphic due to the local uniform convergence, and $g(c) \neq 0$, we conclude that

$$o_c(f) = \sum_{j=1}^{j_0-1} o_c(f_j) + o_c(g) = \sum_{j=1}^{\infty} o_c(f_j).$$

The previous equality also proves that $Z(f) = \bigcup_{j=1}^{\infty} Z(f_j)$. $\square$

In the exercise classes we will show the product formula for the sinus

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big).$$

For the proof we need the logarithmic derivative of infinite products, which will be the last topic of this chapter. Recall that the logarithmic derivative of a holomorphic function $f : U \to \mathbb{C}$ ($f \not\equiv 0$) is by definition the holomorphic function $h : U \setminus Z(f) \to \mathbb{C}$ given by $h = \frac{f'}{f}$. For infinite products the logarithmic derivative has a special structure.

Proposition 3.12. *Let $f_j : U \to \mathbb{C}$ be a sequence of holomorphic functions such that the product $f = \prod_{j=1}^{\infty} f_j : U \to \mathbb{C}$ converges locally normally. Then the logarithmic derivative $\frac{f'}{f} : U \setminus Z(f) \to \mathbb{C}$ is given by*

$$\frac{f'}{f} = \sum_{j=1}^{\infty} \frac{f_j'}{f_j},$$

where the series $\displaystyle\sum_{j=1}^{\infty} \frac{f_j'}{f_j}$ converges locally normally on $U \setminus Z(f)$.

Proof. Note that all functions $h_j = \frac{f_j'}{f_j}$ are holomorphic on $U \setminus Z(f)$. Moreover, we can write

$$f = f_1 \cdot \ldots \cdot f_{n-1} \underbrace{\prod_{j=n}^{\infty} f_j}_{=:g_n},$$

with g_n holomorphic due to local normal convergence and $g_n(z) \neq 0$ for all $z \in U \setminus Z(f)$. By iteration of the standard product rule we then calculate

$$\frac{f'}{f} = \frac{\sum_{j=1}^{n-1} f_j' \prod_{\substack{\ell=1 \\ \ell \neq j}}^{n-1} f_\ell g_n + \prod_{j=1}^{n-1} f_j g_n'.}{\prod_{j=1}^{n-1} f_j g_n} = \sum_{j=1}^{n-1} \frac{f_j'}{f_j} + \frac{g_n'}{g_n}.$$

Combining Lemma 3.9 and Corollary 3.7 (i) we know that g_n converges locally uniformly to 1. From Theorem 1.5 we infer that also $g_n' \to 0$ locally uniformly. In particular, since g_n converges to a non-vanishing function, this implies that the logarithmic derivative $\frac{g_n'}{g_n}$ converges locally uniformly to 0. Thus

$$\frac{f'}{f} = \lim_{n \to +\infty} \sum_{j=1}^{n-1} \frac{f_j'}{f_j} = \sum_{j=1}^{\infty} \frac{f_j'}{f_j} \qquad \text{locally uniformly.}$$

It remains to show that the series converges locally normally on $U \setminus Z(f)$. Fix $z_0 \in U \setminus Z(f)$ and let $r > 0$ be such that $\overline{B_{2r}(z_0)} \subset U \setminus Z(f)$. By Corollary 3.7 (i) the sequence f_j converges locally uniformly on U to 1. Hence we find an index $j_0 \in \mathbb{N}$ such that $|f_j(z)| \geq \frac{1}{2}$ for all $j \geq j_0$ and $z \in B_r(z_0)$. Setting $g_j = f_j - 1$ we conclude that

$$\sum_{j=j_0}^{\infty} \sup_{z \in B_r(z_0)} \left| \frac{f_j'(z)}{f_j(z)} \right| \leq 2 \sum_{j=j_0}^{\infty} \sup_{z \in B_r(z_0)} |g_j'(z)| \leq 4 \sum_{j=j_0}^{\infty} \frac{1}{r} \sup_{z \in \partial B_{2r}(z_0)} |g_j(z)|, \tag{6}$$

where in the last inequality we used the standard Cauchy estimate for derivatives derived from Corollary 0.3 in the form

$$|g_j'(\hat{z})| = \left| \frac{1}{2\pi i} \int_{\partial B_{2r}(z_0)} \frac{g_j(z)}{(z - \hat{z})^2} \, \mathrm{d}z \right| \leq \frac{2}{r} \sup_{z \in \partial B_{2r}(z_0)} |g_j(z)| \qquad \forall \hat{z} \in B_r(z_0).$$

By the local normal convergence of the infinite product the last sum in (6) is finite. This proves the claim. □

Remark 3.13. (i) Proposition 3.12 holds verbatim if we replace local normal convergence by local uniform convergence everywhere. The proof remains unchanged except that we do not need the last argument.

(ii) Even if each $f_j \not\equiv 0$, it can happen that $Z(f) = U$ when U is not connected (take for each connected component an f_j that vanishes only on this component). If instead U is connected, then $Z(f_j)$ is at most a countable set, so that $Z(f)$ can be at most countable.

This was the last result we wanted to prove on infinite products. Next we apply them to prove the celebrated product theorem of Weierstrass.

4. The Weierstrass product theorem

The zeros of a non-constant entire function are always discrete by the identity theorem. In this chapter we study the reverse problem: given a discrete set $S = \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$, does there exist an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $Z(f) = S$ with prescribed multiplicity at each zero. If the set S is finite, say $a_1, \ldots, a_N$ (with multiple occurrences allowed), then the polynomial

$$P(z) = \prod_{j=1}^{N} (z - a_n)$$

satisfies all properties. The Weierstrass product theorem gives an existence result in the infinite case.

In general one cannot expect the convergence of the infinite product

$$\prod_{n=1}^{\infty}\Big(1-\frac{z}{a_n}\Big).$$

Hence we add factors $g_n:\mathbb{C}\to\mathbb{C}\setminus\{0\}$ such that $(1-\frac{z}{a_n})g_n(\frac{z}{a_n})$ is close to 1. Since $|a_n|\to+\infty$ it suffices to note that for $|z|$ small enough we have

$$(1-z)e^{-\log(1-z)}=1.$$

Hence we consider a suitable Taylor-polynomial of the function $z\mapsto-\log(1-z)$ at the origin. Note that for $|z|\leq\frac{1}{2}$ we can write

$$-\log(1-z)=\sum_{k=1}^{\infty}\frac{z^k}{k}.$$

This motivates the definition of the so-called Weierstrass factors given by

$$E_0(z)=1-z,\qquad E_n(z)=(1-z)e^{\sum_{k=1}^{n}\frac{z^k}{k}}.$$

The following estimate turns out to be very useful for proving the Weierstrass product theorem.

Lemma 4.1. *It holds that $|E_n(z)-1|\leq|z|^{n+1}$ for all $z\in B_1(0)$.*

Proof. The claim is trivial for $n=0$. Hence fix $n\geq1$. To reduce notation we set $p_n(z)=\sum_{k=1}^{n}\frac{z^k}{k}$. Then on the one hand

$$(1-z)p_n'(z)=(1-z)\sum_{k=0}^{n-1}z^k=1-z^n$$

and therefore by the product rule

$$E_n'(z)=-e^{p_n(z)}+(1-z)e^{p_n(z)}p_n'(z)=-z^ne^{p_n(z)}.$$

On the other hand, denoting by $\sum_{k=0}^{\infty}a_kz^k$ the Taylor series of E_n at the origin we have that

$$E_n'(z)=\sum_{k=0}^{\infty}ka_kz^{k-1}=-z^ne^{p_n(z)}.$$

The right hand side term has a zero of order n in $z=0$. Hence we conclude that

$$a_k=0\qquad\forall 1\leq k\leq n.$$

Moreover, as the coefficients of the Taylor series of $z\mapsto e^{p_n(z)}$ are all non-negative, we conclude that

$$|a_k|=-a_k\qquad\forall k>n.$$

Since $1=E_n(0)=a_0$ and therefore $0=E_n(1)=1+\sum_{k>n}a_k$, we conclude by Hölder's inequality that

$$|E_n(z)-1|\leq\sum_{k=n+1}^{\infty}|a_k||z|^k\leq\underbrace{\sup_{k>n}|z|^k}_{=|z|^{n+1}}\underbrace{\sum_{k=n+1}^{\infty}|a_k|}_{=1}=|z|^{n+1},$$

where we used that $|z|\leq1$. This concludes the proof. □

With the previous lemma at hand we can now show the Weierstrass product theorem on the complex plane. Note that the existence result remains valid on any open set, but the structure of the function f will be slightly different (cf. the non-mandatory exercise H 7.3).

Theorem 4.2. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers that is discrete in $\mathbb{C}$. For each $n\in\mathbb{N}$ set $o_n:=\#\{k\in\mathbb{N}:a_k=a_n\}$. Assume that $a_n\neq0$ and $o_n<+\infty$ for all $n\in\mathbb{N}$. Then*

$$f(z):=\prod_{n=1}^{\infty}E_n(\tfrac{z}{a_n})=\prod_{n=1}^{\infty}\Big(1-\frac{z}{a_n}\Big)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots+\frac{1}{n}\left(\frac{z}{a_n}\right)^n}$$

converges locally normally and defines an entire function with $Z(f) = \{a_n\}_{n\in\mathbb{N}}$ *and* $o_{a_n}(f) = o_n$ *for all* $n \in \mathbb{N}$.

Remark 4.3. Note that the function $z \mapsto z^k f(z)$ allows to add a zero in $z = 0$ with multiplicity $k \in \mathbb{N}$ to the above result.

Proof of Theorem 4.2. We show that the infinite product defining f converges locally normally on $\mathbb{C}$. To this end, fix a compact set $K \subset \mathbb{C}$. First note that $\lim_{n\to+\infty} |a_n| = +\infty$. Hence there exists $n_0 \in \mathbb{N}$ such that $\left|\frac{z}{a_n}\right| \le \frac{1}{2}$ for all $z \in K$ and $n \ge n_0$. In particular, we can apply Lemma 4.1 to deduce that

$$\sum_{n=n_0}^{\infty} \sup_{z\in K} |E_n(\tfrac{z}{a_n}) - 1| \le \sum_{n=n_0}^{\infty} 2^{-(n+1)} < +\infty.$$

By definition this shows the local normal convergence of the product. Corollary 3.7(iii) then implies that the function f is entire. Moreover, by Lemma 3.11 we know that

$$Z(f) = \bigcup_{n=1}^{\infty} Z\big(E_n\big(\tfrac{\cdot}{a_n}\big)\big) = \{a_n\}_{n\in\mathbb{N}}, \qquad o_{a_n}(f) = \sum_{j=1}^{\infty} o_{a_n}\big(E_n\big(\tfrac{\cdot}{a_n}\big)\big) = o_n,$$

where in the last equality we used that each E_n as a first order zero in $z = 1$. ☐

The Weierstrass product theorem implies the following representation result of entire functions.

Corollary 4.4. *Let* $g : \mathbb{C} \to \mathbb{C}$ *be an entire function such that* $g \not\equiv 0$ *and write its zeros in* $\mathbb{C} \setminus \{0\}$ *as*

$$(\underbrace{a_1, \ldots, a_1}_{o_{a_1}(g) \text{ times}}, \underbrace{a_2, \ldots, a_2}_{o_{a_2}(g) \text{ times}}, \ldots) =: (s_1, s_2, s_2, \ldots) = s$$

Then we can write

$$g(z) = e^{h(z)} z^{o_0(g)} \prod_{n=1}^{\dim(s)} E_n\big(\tfrac{z}{s_n}\big),$$

where $h : \mathbb{C} \to \mathbb{C}$ *is an entire function.*

Proof. Applying the Weierstrass product theorem (or its finite analogue) to the sequence $(s_n)_{n\in\mathbb{N}}$ yields that the function

$$f(z) = z^{o_0(g)} \prod_{n=1}^{\dim(S)} E_n\big(\tfrac{z}{s_n}\big)$$

is entire with $Z(f) = Z(g)$ and $o_z(f) = o_z(g)$ for all $z \in \mathbb{C}$. Hence the quotient g/f has only removable singularities and therefore represents an entire function that never vanishes. It is a well-known result from complex analysis that on the simply connected domain $\mathbb{C}$ this implies that $g/f = e^h$ for some entire function $h : \mathbb{C} \to \mathbb{C}$ (see also Corollary 5.8). This finishes the proof. ☐

5. Picard's little and great theorem

We now come to two celebrated theorems in complex analysis for functions in one variable. The two theorems by Picard provide a very fine description of the image of entire functions (Picard's little theorem) or the image of a neighborhood of an essential singularity (Picard's great theorem). We will see that the little theorem follows (in an even stronger form) from Picard's great theorem. Let us first formulate the two theorems.

Theorem 5.1 (Picard's little theorem)**.** *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a non-constant, entire function. Then* f *assumes each value in* $\mathbb{C}$ *except at most one.*

Theorem 5.2 (Picard's great theorem)**.** *Let* $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ *be holomorphic and let* z_0 *be an essential singularity. Then in each punctured neighborhood of* z_0 f *assumes each value in* $\mathbb{C}$ *infinitely many times except at most one.*

Proof of Theorem 5.1 based on Theorem 5.2. If f is a polynomial then the claim follows by the fundamental theorem of algebra as $z \mapsto f(z) - w$ as a zero for every $w \in \mathbb{C}$, so that f is surjective. If f is not a polynomial, then $z \mapsto f(\frac{1}{z})$ has an essential singularity in $z = 0$ (this follows from the characterization via the principal part of the Laurent series expansion). Hence the claim follows from Theorem 5.2 as the map $z \mapsto \frac{1}{z}$ is one-to-one from $\mathbb{C} \setminus \{0\}$ onto itself. □

Remark 5.3. (i) The above proof shows that if f is not a polynomial then f assumes each value even infinitely many times except at most one.
(ii) Considering the function $z \mapsto e^z$ shows that Theorem 5.1 is sharp.

Next we turn to the proof of Picard's great theorem. On the way we prove several theorems that are interesting on their own. Picard's great theorem will then be a consequence of a strengthened version of Montel's compactness theorem. Let us mention that Picard proved the two theorems by different means.

We start with Bloch's theorem which gives a lower bound on the size of maximal discs contained in the image of non-constant holomorphic functions. In what follows we denote by $\mathcal{H}(\overline{U})$ those functions which are holomorphic in a neighborhood of $\overline{U}$.

Theorem 5.4 (Bloch's theorem)**.** *Let $f \in \mathcal{H}(\overline{B_1(0)})$ be such that $f'(0) = 1$. Then there exists $p \in \mathbb{C}$ such that $B_{\frac{3}{2}-\sqrt{2}}(p) \subset f(B_1(0))$.*

Proof. We divide the proof into three steps. The first two are more general statements.

Step 1: We show that if $U \subset \mathbb{C}$ is a bounded domain, $g \in \mathcal{H}(\overline{U})$ is not constant and $a \in U$ is such that $s = \inf_{z \in \partial U} |g(z) - g(a)| > 0$, then $B_s(g(a)) \subset g(U)$.

Indeed, due to the boundedness of g on U the set $\partial g(U)$ is compact. Hence there exists $w \in \partial g(U)$ such that $\operatorname{dist}(\partial g(U), g(a)) = |w - g(a)|$. We argue that $|w - g(a)| \geq s$ which proves the first step. To this end, note that there exists a sequence $z_n \in U$ such that $g(z_n) \to w$ and without loss of generality also $z_n \to z \in \overline{U}$. Then by continuity $g(z) = w \in \partial g(U)$. By the open mapping theorem it follows that $z \in \partial U$. Hence by definition $|w - g(a)| \geq s$.

Step 2: Next we prove that if $g \in \mathcal{H}(\overline{B_r(a)})$ is not constant and $\sup_{z \in B_r(a)} |g'(z)| \leq 2|g'(a)|$, then $B_R(g(a)) \subset g(B_r(a))$ for $R = (3 - 2\sqrt{2})r|g'(a)|$.

Here comes the argument. Upon considering $z \mapsto g(z + a) - g(a)$ we can assume that $a = g(a) = 0$. Then the function $A(z) = g(z) - g'(0)z$ satisfies

$$A(z) = \int_{[0,z]} g'(\zeta) - g'(0)\,\mathrm{d}\zeta,$$

so that by the definition of the path-integral we have the bound

$$|A(z)| \leq \int_0^1 |g'(tz) - g'(0)||z|\,\mathrm{d}t.$$

In order to bound the difference in the integrand we express it by Cauchy's integral formula as

$$g'(v) - g'(0) = \frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{g'(\zeta)}{\zeta - v} - \frac{g'(\zeta)}{\zeta}\,\mathrm{d}\zeta = \frac{v}{2\pi i} \int_{\partial B_r(0)} \frac{g'(\zeta)}{\zeta(\zeta - v)}\,\mathrm{d}\zeta \qquad \forall v \in B_r(0),$$

so that still for $v \in B_r(0)$ we have

$$|g'(v) - g'(0)| \leq \frac{|v|}{r - |v|} \sup_{z \in B_r(0)} |g'(z)|.$$

Combined with our assumption this yields the following bound on $A(z)$:

$$|A(z)| \leq \int_0^1 \frac{|tz|}{r - |tz|} \sup_{z \in B_r(0)} |g'(z)||z|\,\mathrm{d}t \leq \frac{1}{2}\frac{|z|^2}{r - |z|} \sup_{z \in B_r(0)} |g'(z)| \leq \frac{|z|^2}{r - |z|}|g'(0)|.$$

Since also $|A(z)| \geq |g'(0)||z| - |g(z)|$, we deduce that for all $z \in B_r(0)$ it holds that

$$|g(z)| \geq \left(|z| - \frac{|z|^2}{r - |z|}\right)|g'(0)|.$$

In order to apply Step 1 in the most efficient way we consider the sphere $\partial B_{\rho^*}(0)$ where the term in brackets gets maximal. With elementary analysis one can show that the real-valued function $\rho \mapsto \rho - \rho^2(r-\rho)^{-1}$ takes its maximum on $(0,r)$ at $\rho^* = (1-\frac{1}{2}\sqrt{2})r$ with value $(3-2\sqrt{2})r$. Hence applying Step 1 with $U = B_{\rho^*}(0)$ and $a = g(a) = 0$ yields $B_R(0) \subset g(B_r(0))$ with R as claimed.

Step 3: Conclusion.

To the function $f \in \mathcal{H}(\overline{B_1(0)})$ we associate the function $z \mapsto h(z) := |f'(z)|(1-|z|)$, which is continuous on $\overline{B_1(0)}$. Since by assumption $f'(0) = 1$ it follows that the maximum of h on $\overline{B_1(0)}$ is assumed at some point $p \in B_1(0)$ with $M := h(p) \geq |f'(0)| = 1$. Setting $r = \frac{1}{2}(1-|p|)$ we have $M = 2r|f'(p)|$ and $B_r(p) \subset B_1(0)$. Moreover, note that for $z \in B_r(p)$ it holds that

$$|z| \leq |p| + r = 1 - r,$$

or equivalently $(1-|z|) \geq r$. Using the maximality $|f'(z)|(1-|z|) \leq 2r|f'(p)|$ we conclude that $|f'(z)| \leq 2|f'(p)|$ for all $z \in B_r(p)$, so that Step 2 implies that $B_R(f(p)) \subset f(B_1(0))$ for

$$R = (3-2\sqrt{2})r|f'(p)| \geq (\tfrac{3}{2} - \sqrt{2})M \geq (\tfrac{3}{2} - \sqrt{2})$$

as claimed ☐

Bloch's theorem might seem quite restrictive as formulated only on the unit disc. But there are some straightforward consequences.

Corollary 5.5. *If $f : U \to \mathbb{C}$ is holomorphic and $f'(c) \neq 0$ at a point $c \in U$, then $f(U)$ contains discs of every radius $(\frac{3}{2} - \sqrt{2})s|f'(c)|$ for $0 < s < \operatorname{dist}(c, \partial U)$. In particular, if $f : \mathbb{C} \to \mathbb{C}$ is entire and non-constant, then $f(\mathbb{C})$ contains discs of arbitrarily large radii.*

Proof. See Exercise H 8.1. ☐

Both Picard's little and Picard's great theorem deal with functions that omit two values. Hence we need to study them more in detail. Before we continue, we recall the notion of simply connected (open) sets.

Definition 5.6. Let $G \subset \mathbb{C}$ be an open set. We say that G is simply connected if it is path-connected and every closed curve $\gamma \subset G$ can be contracted in G to a point, that is, for every continuous curve $\gamma : [0,1] \to G$ with $\gamma(0) = \gamma(1)$ there exists a point $z_0 \in G$ and a continuous map $H : [0,1] \times [0,1] \to G$ such that

(i) $H(0,t) = \gamma(t)$ $\quad \forall t \in [0,1]$;
(ii) $H(1,t) = z_0$ $\quad \forall t \in [0,1]$;
(iii) $H(s,0) = H(s,1)$ $\quad \forall s \in [0,1]$.

We will heavily rely on the fact that on simply connected domains every holomorphic function has a primitive. We omit its proof as it is treated in basic courses on complex analysis.

Theorem 5.7. *Let $G \subset \mathbb{C}$ be a simply connected domain and let $f : G \to \mathbb{C}$ be holomorphic. Then there exists a holomorphic function $F : G \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in G$.* ☐

Based on that theorem we can show that on simply connected domains there always exist holomorphic logarithms and n^{th}-roots.

Corollary 5.8. *Let $G \subset \mathbb{C}$ be a simply connected domain and let $f : G \to \mathbb{C} \setminus \{0\}$. Then there exists a holomorphic function $\log(f) : G \to \mathbb{C}$ such that $\exp(\log(f)) = f$. Moreover, for each $n \in \mathbb{N}$ there exists a holomorphic function $\sqrt[n]{f} : G \to \mathbb{C}$ such that $(\sqrt[n]{f})^n = f$.*

Proof. Consider the logarithmic derivative $h : G \to \mathbb{C}$ defined by $h = f'/f$, which is holomorphic on G. Choose a primitive $H : G \to \mathbb{C}$ such that for some $z_0 \in G$ we have $H(z_0) = \log(f(z_0))$. Then by the product rule we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(f(z)\exp(-H(z))\right) = f'(z)\exp(-H(z)) - f(z)\exp(-H(z))\frac{f'(z)}{f(z)} = 0.$$

Since also $\exp(H(z_0)) = f(z_0)$ we deduce from the path-connectedness of G that $\exp(H(z)) = f(z)$ for all $z \in G$. Thus setting $\log(f) = H$ shows the first assertion.

In order to prove the second statement, it suffices to define $\sqrt[n]{f} : G \to \mathbb{C}$ by $\sqrt[n]{f}(z) = \exp(\frac{1}{n}H(z))$. □

Now we are in a position to show the following auxiliary result on holomorphic functions that omit two values.

Lemma 5.9. *Let $G \subset \mathbb{C}$ be a simply connected domain and let $f : G \to \mathbb{C}$ be holomorphic such that $\{-1, 1\} \cap f(G) = \emptyset$. Then there exists a holomorphic function $h : G \to \mathbb{C}$ such that*

$$f = \cos(h).$$

Proof. Note that the function $z \mapsto 1 - f(z)^2$ never vanishes on G. Hence by Corollary 5.8 there exists a holomorphic square root $g = \sqrt{1 - f^2}$, which satisfies in particular

$$(f + ig)(f - ig) = f^2 + g^2 = 1. \tag{7}$$

Thus $(f + ig)$ has no zeros in G and therefore we can write $(f + ig) = e^{ih}$ for some holomorphic function $h : G \to \mathbb{C}$. Then by (7) it holds that $(f - ig) = e^{-ih}$, so that $f = \frac{1}{2}(e^{ih} + e^{-ih}) = \cos(h)$ as claimed. □

With this lemma we can prove the following crucial proposition.

Proposition 5.10. *Let $G \subset \mathbb{C}$ be a simply connected domain and let $f : G \to \mathbb{C}$ be holomorphic such that $\{0, 1\} \cap f(G) = \emptyset$. Then there exists a holomorphic function $h : G \to \mathbb{C}$ such that*

$$f = \frac{1}{2}\left(1 + \cos(\pi \cos(\pi h))\right).$$

If $\widetilde{h} : G \to \mathbb{C}$ is any holomorphic function satisfying the above equation, then $\widetilde{h}(G)$ contains no disc of radius larger or equal than 1.

Proof. First note that the function $2f - 1$ omits the values -1 and 1, so that by Lemma 5.9 we find a holomorphic function $h_1 : G \to \mathbb{C}$ such that $2f - 1 = \cos(\pi h_1)$. Observe further that h_1 must omit all integer values. Hence again by Lemma 5.9 we can write $h_1 = \cos(\pi h)$ for some holomorphic function $h : G \to \mathbb{C}$. The first claim then follows by rearranging terms.

Now let $\widetilde{h} : G \to \mathbb{C}$ be any such function. Define the grid-like set

$$\mathcal{L} = \{m \pm i\pi^{-1}\log(n + \sqrt{n^2 - 1}) : \ m \in \mathbb{Z}, n \in \mathbb{N} \setminus \{0\}\}.$$

We shall prove that $\mathcal{L} \cap \widetilde{h}(G) = \emptyset$. Indeed, for $\hat{z} := m \pm i\pi^{-1}\log(n + \sqrt{n^2 - 1}) \in \mathcal{L}$ we have that

$$\begin{aligned}
\cos(\pi\hat{z}) =& \frac{1}{2}(e^{i\pi\hat{z}} + e^{-i\pi\hat{z}}) = \frac{1}{2}(-1)^m\left((n + \sqrt{n^2 - 1})^{\mp} + (n + \sqrt{n^2 - 1})^{\pm}\right) \\
=& \frac{1}{2}(-1)^m \frac{n^2 + n^2 - 1 + 2n\sqrt{n^2 - 1} + 1}{n + \sqrt{n^2 - 1}} = (-1)^m n.
\end{aligned}$$

Thus $\cos(\pi\cos(\pi\hat{z})) = \pm 1$ for all $\hat{z} \in \mathcal{L}$. Since $f(G) \cap \{0, 1\} = \emptyset$ we conclude that $\widetilde{h}(G) \cap \mathcal{L} = \emptyset$ as claimed. It remains to estimate the grid-size of $\mathcal{L}$. Note that the 'vertical' distance between neighboring grid points is bounded by

$$\begin{aligned}
|\log(n + 1 + \sqrt{(n + 1)^2 - 1}) - \log(n + \sqrt{n^2 - 1})| =& \left|\log\left(\frac{1 + n^{-1} + \sqrt{1 + 2n^{-1}}}{1 + \sqrt{1 - n^{-2}}}\right)\right| \\
\leq& \log(1 + n^{-1} + \sqrt{1 + 2n^{-1}}) \leq \log(2 + \sqrt{3}) < \pi,
\end{aligned}$$

where we assumed without loss of generality that the two points are in the upper half plane (note that for $n = 1$ the points in $\mathcal{L}$ are on the real line so that there are no neighboring points in different half-planes). The 'horizontal' distance is exactly 1. Hence for every $z \in \mathbb{C}$ there exists $\hat{z} \in \mathcal{L}$ such that $|\mathrm{Re}(z) - \mathrm{Re}(\hat{z})| \leq 1/2$ and $|\mathrm{Im}(z) - \mathrm{Im}(\hat{z})| < 1/2$. Hence $|z - \hat{z}| < 1$. So every disc of radius 1 in $\mathbb{C}$ intersects $\mathcal{L}$. Hence $\widetilde{h}(G)$ cannot contain a disc of radius larger or equal than one. □

With Proposition 5.10 one obtains Picard's little theorem (cf. exercise H 9.1). Towards the proof of Picard's great theorem it helps us to prove Schottky's theorem which controls the growth of functions omitting the two values 0 and 1.

Denote by $\beta > 0$ a constant for which Bloch's theorem holds (e.g. $\beta = (3/2 - \sqrt{2})$). Let us define the positive function $L : (0,1) \times (0,+\infty) \to \mathbb{R}_+$ by

$$L(\theta, r) := \exp\left(\pi \exp\left(\pi\left(3 + 2r + \frac{\theta}{\beta(1-\theta)}\right)\right)\right)$$

Then we have the following result:

Theorem 5.11 (Schottky's theorem). *Let $f \in \mathcal{H}(\overline{B_1(0)})$ be such that $|f(0)| \le r$ and $\{0,1\} \cap f(\overline{B_1(0)}) = \emptyset$. Then*

$$|f(z)| \le L(\theta, r) \qquad \forall |z| \le \theta,\ 0 < \theta < 1.$$

Proof. We divide the proof into several steps.
Step 1: We first show that if $\cos(\pi a) = \cos(\pi b)$, then $b = \pm a + 2n$ for some $n \in \mathbb{Z}$ and that for every $w \in \mathbb{C}$ there exists $v \in \mathbb{C}$ such that $\cos(\pi v) = w$ and $|v| \le 1 + |w|$. The first claim follows from the formula $\cos(\pi a) - \cos(\pi b) = -2\sin(\frac{\pi}{2}(a+b))\sin(\frac{\pi}{2}(a-b))$ and from $\{\sin = 0\} = \pi\mathbb{Z}$. Since $z \mapsto \cos(z)$ is surjective onto $\mathbb{C}$, for every $w \in \mathbb{C}$ we can thus find $v \in \mathbb{C}$ with $\mathrm{Re}(v) \in [-1,1]$ and $\cos(\pi v) = \omega$. Since

$$|w|^2 = \cos^2(\pi \mathrm{Re}(v)) + \sinh^2(\pi \mathrm{Im}(v))$$

and $\sinh^2(x) \ge x^2$ for all $x \in \mathbb{R}$ (proof by power series representation of sinh for $x \ge 0$), we deduce that

$$|v| = \sqrt{\mathrm{Re}(v)^2 + \mathrm{Im}(v)^2} \le \sqrt{1 + |w|^2/\pi^2} \le 1 + |w|.$$

Step 2: There exists a function $g \in \mathcal{H}(\overline{B_1(0)})$ such that

(i) $f = \frac{1}{2}(1 + \cos(\pi\cos(\pi g)))$ with $|g(0)| \le 3 + 2|f(0)|$;
(ii) $|g(z)| \le |g(0)| + \theta/(\beta(1-\theta))$ for all $|z| \le \theta$, $0 < \theta < 1$.

Indeed, by Lemma 5.9 we find a function $\widetilde{F} \in \mathcal{H}(\overline{B_1(0)})$ such that $2f - 1 = \cos(\pi\widetilde{F})$. Due to Step 1 there exists $b \in \mathbb{C}$ such that $\cos(\pi b) = 2f(0) - 1$ and $|b| \le 1 + |2f(0) - 1| \le 2 + 2|f(0)|$. Moreover, again by Step 1 $b = \pm\widetilde{F}(0) + 2k$ with $k \in \mathbb{Z}$. Define $F = \pm\widetilde{F} + 2k$, so that $F \in \mathcal{H}(\overline{B_1(0)})$. Then $2f - 1 = \cos(\pi F)$ and $F(0) = b$. Since F omits all integer values there exists $\widetilde{g} \in \mathcal{H}(\overline{B_1(0)})$ such that $F = \cos(\pi\widetilde{g})$. Using one more time Step 1 we find $a \in \mathbb{C}$ such that $\cos(\pi a) = b$ and $|a| \le 1 + |b| \le 3 + 2|f(0)|$. By construction $\cos(\pi a) = \cos(\pi\widetilde{g}(0))$, so that we can again define $g = \pm\widetilde{g} + 2m \in \mathcal{H}(\overline{B_1(0)})$ for some $m \in \mathbb{Z}$ such that $g(0) = a$ and $F = \cos(\pi g)$. Then $f = \frac{1}{2}(1 + \cos(\pi\cos(\pi g)))$ and $|g(0)| = |a| \le 3 + 2|f(0)|$ as claimed in (i).

In order to show (ii), note that by Proposition 5.10 $g(B_1(0))$ contains no disc of radius larger or equal than 1. Since $\mathrm{dist}(z, \partial B_1(0)) \ge (1-\theta)$ for all $|z| \le \theta$, the generalized Bloch theorem (cf. exercise H 8.1) implies that $\beta(1-\theta)|g'(z)| \le 1$ for all $|z| \le \theta$. Rearranging terms yields $|g'(z)| \le (\beta(1-\theta))^{-1}$. Thus by the fundamental theorem of calculus

$$\begin{aligned}|g(z)| &\le |g(z) - g(0)| + |g(0)| \le \int_{[0,z]} |g'(\zeta)|\,\mathrm{d}\zeta + |g(0)| \\ &\le \frac{|z|}{\beta(1-\theta)} + |g(0)| \le \frac{\theta}{\beta(1-\theta)} + |g(0)|\end{aligned}$$

for all $|z| \le \theta$.
Step 3: Conclusion. We finish the proof by noting that $|\cos(\omega)| \le e^{|w|}$ (proof via power series) and $\frac{1}{2}|1 + \cos(\omega)| \le e^{|w|}$ (triangle inequality). Indeed, using those bounds and properties (i) and (ii) of Step 2 we deduce that

$$|f(z)| \le \exp(\pi\exp(\pi|g(z)|)) \le \exp(\pi\exp(\pi(3 + 2|f(0)| + \theta(\beta(1-\theta))^{-1}))) \le L(\theta, r),$$

where in the last estimate we used that $|f(0)| \le r$. □

Schottky's theorem allows to prove a sharpened version of Montel's compactness theorem that reads as follows:

Theorem 5.12 (Sharpened version of Montel's theorem)**.** *Let $D \subset \mathbb{C}$ be a domain and*

$$\mathcal{F} := \{f : D \to \mathbb{C} \text{ holomorphic}, \{0,1\} \cap f(D) = \emptyset\}.$$

Let $\{f_n\}_{\in\mathbb{N}} \subset \mathcal{F}$. Then either $\{f_n\}$ contains a subsequence that converges locally uniformly to some holomorphic function $f : D \to \mathbb{C}$ or the whole sequence $|f_n|$ converges locally uniformly to $+\infty$.

Remark 5.13. The local uniform convergence to $+\infty$ is defined by requiring that $1/f_n$ converges locally uniformly to 0.

Proof of Theorem 5.12. We divide the proof into several steps.
Step 1: For $z_0 \in D$ and $r \in (0, +\infty)$ define $\mathcal{F}(z_0, r) := \{f \in \mathcal{F} : |f(z_0)| \leq r\}$. We argue that there exists a neighborhood of z_0 on which $\mathcal{F}(z_0, r)$ is equibounded.

Indeed, choose $\delta > 0$ such that $\overline{B_{2\delta}(z_0)} \subset D$. Applying Schottky's Theorem 5.11 to each function $z \mapsto f(2\delta z + z_0) \in \mathcal{H}(\overline{B_1(0)})$ for $f \in \mathcal{F}(z_0, r)$ we infer that

$$\sup_{f \in \mathcal{F}(z_0,r)} \sup_{z \in B_\delta(z_0)} |f(z)| \leq L(1/2, r).$$

This proves the first step.
Step 2: The family $\mathcal{F}(z_0, r)$ is locally bounded in D.

To prove this claim, first note that $U := \{z \in D : \mathcal{F}(z_0, r) \text{ is equibounded in a neighborhood of } z\}$ is open in D. Moreover $z_0 \in U$ by Step 1. We argue that U is also closed in D. Then connectedness of D implies that $U = D$ and the claim of Step 2 follows by a covering argument. Let $w \in \partial U \cap D$ be such that $\mathcal{F}(z_0, r)$ is not bounded in a neighborhood of w. Then by Step 1 we know that there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(z_0, r)$ such that $\lim_n |f_n(w)| = +\infty$. Define $g_n = 1/f_n$ (which is well-defined), so that $\lim_n g_n(w) = 0$. Hence $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(w, R)$ for some suitable $0 < R < +\infty$. Thus by Step 1 the sequence $\{g_n\}$ is bounded in a neighborhood of w and therefore by Montel's Theorem 1.7 we can pass to a subsequence that converges uniformly in a disc $B_r(w)$ to some holomorphic function $g : B_r(w) \to \mathbb{C}$. Since all g_n have no zeros and $g(w) = 0$, it follows from Corollary 1.6 that $g \equiv 0$. Since $B_r(w) \cap U \neq \emptyset$ we conclude that $\limsup_n |f_n(z)| = +\infty$ also for some $z \in U$, which gives a contradiction.
Step 3: Conclusion.

Fix $z_0 \in D$ and let $\{f_n\} \subset \mathcal{F}$ be a sequence. If infinitely many $f_n \in \mathcal{F}(z_0, 1)$ then the claim follows from Step 2 and Montel's theorem 1.7. If only finitely many f_n belong to $\mathcal{F}(z_0, 1)$, then infinitely many $g_n = 1/f_n$ belong to $\mathcal{F}(z_0, 1)$. By Step 2 we conclude that either all subsequences g_n converge locally uniformly to $g \equiv 0$ or along a subsequence g_n converges to a non-zero limit which has then no zero at all by Corollary 1.6. In the second case also f_n converges locally uniformly along that subsequence. In the first case $|f_n| \to +\infty$ locally uniformly as claimed. ◻

Now we can finally prove Picard's great theorem.

Proof of Theorem 5.2. We have seen in exercise H 9.3 the remarkable statement that Picard's great theorem is equivalent to prove that given a holomorphic function $f : B_1(0) \setminus \{0\} \to \mathbb{C} \setminus \{0, 1\}$ either f or $1/f$ is bounded in a neighborhood of the origin.

Consider the sequence of holomorphic functions $f_n(z) = f(z/n) : B_1(0) \setminus \{0\} \to \mathbb{C} \setminus \{0, 1\}$. By Theorem 5.12 there exists a subsequence f_{n_k} such that either f_{n_k} or $1/f_{n_k}$ is locally uniformly bounded on $B_1(0) \setminus \{0\}$. In the first case there exists $0 < C < +\infty$ such that

$$\sup_k |f(z/n_k)| \leq C \qquad \forall |z| = \frac{1}{2}.$$

Hence by the maximum principle $|f(z)| \leq C$ on each annulus $A_k = \{z \in \mathbb{C} : \frac{1}{2n_{k+1}} \leq |z| \leq \frac{1}{2n_k}\}$. Since

$$A := \bigcup_{k\in\mathbb{N}} A_k$$

is a (punctured) neighborhood of the origin, we conclude that f is bounded in a neighborhood of the origin. The case when $1/f_{n_k}$ is locally bounded can be treated the same way and hence we conclude the proof. ◻

6. The Riemann mapping theorem

In this chapter we will prove one of the main theorems in complex analysis. The Riemann mapping theorem classifies all sets that are biholomorphically equivalent to the open unit disc $B_1(0)$. Here two sets $U_1, U_2 \subset \mathbb{C}$ are said to be biholomorphically equivalent if there exists a bijective holomorphic map $f : U_1 \to U_2$ such that the inverse map $f^{-1} : U_2 \to U_1$ is also holomorphic. It follows from the definition that biholomorphic equivalence of open sets is an equivalence relation.

In order to study the family of sets which are biholomorphically equivalent to the unit disc $B_1(0)$, first note that by Liouville's theorem $\mathbb{C}$ cannot belong to that class. Moreover, as biholomorphic functions are in particular homeomorphisms, all sets belonging to that class share the same topological invariances. In particular, such sets have to be path-connected (as the unit disc) and also simply connected (cf. Definition 5.6).

Simply connected sets in $\mathbb{C}$ can be formally described as having no holes. The surprising fact of the Riemann mapping theorem is that this topological restriction already ensures biholomorphic equivalence to the unit disc. In particular, no smoothness of the boundary is required.

In the proof of the Riemann mapping theorem we will use the fact that injective holomorphic functions are already biholomorphic onto their image. For the sake of completeness we include the proof.

Lemma 6.1. *Let $f : U \to \mathbb{C}$ be holomorphic and injective. Then $f'(z_0) \neq 0$ for all $z_0 \in U$ and the inverse function $f^{-1} : f(U) \to \mathbb{C}$ is also holomorphic.*

Proof. Assume by contradiction that $f'(z_0) = 0$ for some $z_0 \in U$. Upon considering the function $z \mapsto f(z + z_0) - f(z_0)$ we may assume that $z_0 = f(z_0) = f'(z_0) = 0$. Since f is injective it is not constantly zero on any open set, so there exists a minimal $k \in \mathbb{N}$ such that $f^{(k)}(z_0) \neq 0$. Hence we can write

$$f(z) = \sum_{n=k}^{\infty} a_n z^n = z^k \sum_{n=0}^{\infty} a_{k+n} z^n =: z^k g(z),$$

with $g : U \to \mathbb{C}$ holomorphic and $g(0) \neq 0$. Since g is in particular continuous, there exists $r > 0$ such that $B_r(0) \subset U$ and $\inf_{z \in B_r(0)} |g(z)| > 0$. Since $B_r(0)$ is a simply connected domain, we can apply Corollary 5.8 to infer that there exists a holomorphic function $h : B_r(z_0) \to \mathbb{C}$ such that $h^k = g$. Then for all $z \in B_r(0)$ we can write

$$f(z) = (zh(z))^k$$

Note that the function $z \mapsto zh(z)$ is non-constant and holomorphic. Hence by the open mapping theorem there exists $r_1 > 0$ and $z_1, z_2 \in B_r(0)$ such that $z_1 h(z_1) = r_1$ and $z_2 h(z_2) = r_1 \exp(2\pi i/k)$. This contradicts the injectivity of f since $f(z_1) = f(z_2)$.

Moreover, by the open mapping theorem it follows that the set $f(U)$ is open. Since $f'(z_0) \neq 0$ for all $z \in U$ the inverse function theorem yields that the inverse map is differentiable as a function of $\mathbb{R}^2$ to $\mathbb{R}^2$. Since the differential of f is pointwise a scalar multiple of a rotation by the Cauchy-Riemann equations the differential of the inverse has the same structure. Hence it satisfies the Cauchy-Riemann equations, too. We conclude that the inverse map is holomorphic. □

The remainder of this chapter will be about the proof of the Riemann mapping theorem. In the proof we will apply the Schwarz lemma which we recall here.

Lemma 6.2 (Schwarz Lemma)**.** *Let $f : B_1(0) \to \overline{B_1(0)}$ be a holomorphic function such that $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in B_1(0)$ and $|f'(0)| \leq 1$. Moreover, if any of the two is an equality (for some $z \in B_1(0) \setminus \{0\}$) then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.*

Proof. Consider the decomposition $f(z) = zg(z)$. Due to the assumptions we know that $g : B_1(0) \to \mathbb{C}$ is holomorphic and $g(0) = f'(0)$. For $0 < r < 1$ and $z \in \partial B_r(0)$ we have

$$|g(z)| \leq \frac{|f(z)|}{|z|} \leq \frac{1}{r}.$$

Due to the maximum principle this inequality holds true for all $z \in B_r(0)$. Letting $r \uparrow 1$ yields that $|g(z)| \leq 1$ for all $z \in B_1(0)$. This implies $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If one of the two is an equality we deduce that $|g(z_0)| = 1$ for some $z_0 \in B_1(0)$. Again by the maximum principle it follows that g is constant. Hence $g(z) = a$ for some $a \in \mathbb{C}$ with $|a| = 1$. This yields the claim. $\square$

Theorem 6.3 (Riemann mapping theorem). *Let $G \subsetneq \mathbb{C}$ be a simply connected domain. Then there exists a biholomorphic map $f : G \to B_1(0)$.*

Proof. Due to Lemma 6.1 it suffices to find a bijective holomorphic map $f : G \to B_1(0)$. We will prove the existence of such a map in three steps.

(1) We show the existence of an injective holomorphic map $g : G \to B_1(0)$ with $0 \in g(G)$. This allows us to assume that $G \subset B_1(0)$ and $0 \in G$;
(2) For an injective map $f : G \subset B_1(0) \to B_1(0)$ with $f(0) = 0$ we show that surjectivity is ensured by the maximality of $|f'(0)|$;
(3) We find a bijective function by maximizing $|f'(0)|$ under all injective holomorphic functions such that $f(0) = 0$.

The desired function can then be obtained as the composition $f \circ g$.

Step 1: Assume for the moment that the complement $\mathbb{C} \setminus G$ contains an open ball $B_{2r}(z_0)$. Then the map $g_1 : G \to B_1(0)$ given by $g_1(z) = r(z - z_0)^{-1}$ is well-defined, holomorphic and injective. Let $z_1 \in g(G) \subset \overline{B_{1/2}(0)}$. Then the map $g(z) = \frac{1}{2}(g_1(z) - z_1)$ is still injective, holomorphic and $0 \in g(G) \subset B_1(0)$. This gives the desired map of step 1. However, in general we cannot assume that $\mathbb{C} \setminus G$ contains an open ball. Therefore we have to transform it via an injective, holomorphic function. The idea is to use a holomorphic square-root. By assumption there exists $z_0 \in \mathbb{C} \setminus G$. Then the function $z \mapsto z - z_0$ never vanishes on G. Hence by Corollary 5.8 there exists a holomorphic function $G \ni z \mapsto \sqrt{z - z_0}$. We claim that this square-root is injective. Indeed, if $\sqrt{z_1 - z_0} = \sqrt{z_2 - z_0}$, then by definition

$$z_1 - z_0 = \sqrt{z_1 - z_0}^2 = \sqrt{z_2 - z_0}^2 = z_2 - z_0,$$

which implies that $z_2 = z_1$. Moreover, we argue that $\mathbb{C} \setminus \sqrt{G - z_0}$ contains an interior point. By the open mapping theorem there exists $\hat{z} \neq 0$ and $r > 0$ such that $B_{2r}(\hat{z}) \subset \sqrt{G - z_0}$. We claim that $-B_{2r}(\hat{z}) \subset \mathbb{C} \setminus \sqrt{G - z_0}$. Indeed, assume that there exists $z_1, z_2 \in G$ such that $\sqrt{z_1 - z_0} = -\sqrt{z_2 - z_0}$. Taking the square yields $z_1 = z_2$ which is only possible if $z_1 = z_0$. This contradicts the fact that $z_0 \notin G$. Thus we are in a position to apply the first part of this step and from now on we can assume that $0 \in G \subset B_1(0)$. Here we also used that the image of G under an injective holomorphic function is still simply connected (see exercise H 10.1).

Step 2: We claim that if $0 \in G \subset B_1(0)$ and $f : G \to B_1(0)$ is injective, holomorphic, satisfies $f(0) = 0$, but fails to be surjective, then there exists a holomorphic, injective function $\widetilde{f} : G \to B_1(0)$ with $\widetilde{f}(0) = 0$ and $|\widetilde{f}'(0)| > |f'(0)|$.

As a first step we note that for any $z_0 \in B_1(0)$ the map $\varphi_{z_0} : B_1(0) \to \mathbb{C}$ defined by

$$\varphi_{z_0}(z) = \frac{z - z_0}{1 - \overline{z_0} z}$$

is a biholomorphic map onto $B_1(0)$ (cf. exercise H 8.2). We assume that there exists $z_0 \in B_1(0) \setminus f(G)$. Then $\varphi_{z_0} \circ f : G \to B_1(0)$ satisfies $\varphi_{z_0}(f(z)) \neq 0$ for all $z \in G$. By Corollary 5.8 we can define a holomorphic square-root of this map. Then the function $\sqrt{\varphi_{z_0} \circ f}$ is holomorphic, injective and $z_1 := \sqrt{\varphi_{z_0}(f(0))} = \sqrt{-z_0} \in B_1(0)$. We define a competitor for f as

$$\widetilde{f} := \varphi_{z_1} \circ \sqrt{\varphi_{z_0} \circ f} : G \to B_1(0),$$

which is injective, holomorphic and satisfies $\widetilde{f}(0) = 0$. Moreover, setting $h : B_1(0) \to B_1(0)$ as $h = \varphi_{z_0}^{-1} \circ (\varphi_{z_1}^{-1})^2$ it follows that h is holomorphic, satisfies $h(0) = 0$ and $h \circ \widetilde{f} = f$. By the Schwarz Lemma we know that $|h'(0)| < 1$ since h is no pure rotation (not even injective). Hence the chain rule implies

$$|f'(0)| = |h'(\widetilde{f}(0))\widetilde{f}'(0)| = |h'(0)\widetilde{f}'(0)| < |\widetilde{f}'(0)|$$

as claimed.

Step 3: In order to construct a bijective holomorphic function $f : G \to B_1(0)$ it is enough to find a solution to the following optimization problem:

$$\sup\{|f'(0)| : f : G \to B_1(0) \text{ holomorphic and injective, } f(0) = 0\}.$$

Indeed, by Step 2 such a function is surjective and therefore satisfies the claimed properties. We have already seen in exercise H 3.3 that a maximizer for the above extremal problem exists provided the class of competitors is not empty. Since we reduced the analysis to the case that $0 \in G \subset B_1(0)$ the function $z \mapsto z$ is admissible. This concludes the proof. □

Remark 6.4. One can prove that under the assumption that $f(z_0) = 0$ and $f'(z_0) \in (0, +\infty)$ for some $z_0 \in G$ the function f is unique. Indeed, suppose there is another function $g : G \to B_1(0)$ with the given properties. Then $h := f \circ g^{-1} : B_1(0) \to B_1(0)$ satisfies $h(0) = 0$ and $h'(0) \in (0, +\infty)$. The inverse map h^{-1} satisfies the same properties. Applying the Schwarz Lemma to both functions yields $|h(z)| = |z|$ for all $z \in B_1(0)$. Hence again by the Schwarz Lemma $h(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$. Then from $h'(0) = a \in (0, +\infty)$ we infer that $a = 1$. This proves uniqueness.

We will not discuss whether the map f given by the Riemann mapping theorem can be extended to the boundary. Let us just mention that such an extension requires some regularity of the boundary of G.

7. Holomorphic functions on the Riemann sphere

In many situations it is convenient to allow the value ∞ either in the domain or the image of functions (cf. meromorphic functions, the sharpened version of Montel's theorem). This can be done via the one-point compactification of $\mathbb{C}$, denoted by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Open sets in $\widehat{\mathbb{C}}$ are defined to be exactly those sets $\widehat{U} \in \widehat{\mathbb{C}}$ such that

(i) $\widehat{U}$ is open in $\mathbb{C}$ if $\infty \notin \widehat{U}$;
(ii) $\widehat{U} \setminus \{\infty\} = \mathbb{C} \setminus K$ for some compact set $K \subset \mathbb{C}$ if $\infty \in \widehat{U}$.

With this topology one can prove that $\widehat{\mathbb{C}}$ is a compact metrizable space which is homoeomorphic to the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ (cf. exercise H 11.1). Moreover, a sequence $\{z_n\}_{n\in\mathbb{N}}$ converges to ∞ if and only if $1/z_n$ converges to zero, while the convergence on $\widehat{\mathbb{C}} \setminus \{\infty\} = \mathbb{C}$ remains unaffected. In this sense we do not distinguish in which direction the sequence approaches infinity. Hence in what follows we tacitly set $1/\infty = 0$ and $1/0 = \infty$. Then we have the following definition of holomorphic functions $f : \widehat{U} \to \widehat{\mathbb{C}}$:

Definition 7.1. Let $\widehat{U} \subset \widehat{\mathbb{C}}$ be open and let $f : \widehat{U} \to \widehat{\mathbb{C}}$ be continuous. Then f is called complex differentiable in $z_0 \in \widehat{U}$ if

(i) f is complex differentiable in the usual sense if $z_0, f(z_0) \in \mathbb{C}$;
(ii) $g(z) = f(\frac{1}{z})$ is complex differentiable in 0 if $z_0 = \infty$ and $f(z_0) \in \mathbb{C}$;
(iii) $g(z) = \frac{1}{f(z)}$ is complex-differentiable in z_0 if $z_0 \in \mathbb{C}$ and $f(z_0) = \infty$;
(iv) $g(z) = \frac{1}{f(\frac{1}{z})}$ is complex differentiable in 0 if $z_0 = f(z_0) = \infty$.

f is called holomorphic on $\widehat{U}$ if f is complex differentiable in every point $z_0 \in \widehat{U}$.

Remark 7.2. In the theory of Riemann surfaces the above notion corresponds to holomorphic functions on a manifold since $z \mapsto z$ and $z \mapsto 1/z$ are charts for the one-dimensional complex manifold $\widehat{\mathbb{C}}$.

Similar to the case of domains $D \subset \mathbb{C}$ the identity theorem also holds for holomorphic functions $f : \widehat{D} \to \widehat{\mathbb{C}}$ (domains $\widehat{D}$ in $\widehat{\mathbb{C}}$ are defined to be open, path-connected subsets).

Theorem 7.3 (Identity theorem)**.** *Let $\widehat{D} \subset \widehat{\mathbb{C}}$ be a domain and let $f, g : \widehat{D} \to \widehat{\mathbb{C}}$ be holomorphic. If the set $\{f = g\}$ has an accumulation point in $\widehat{D}$, then $f = g$.*

Proof. Let us define $S := \{z \in \widehat{D} : f = g \text{ in a neighborhood of } z\}$. We argue that $S = \widehat{D}$. First note that S is open. Next we show that $S \neq \emptyset$. To this end, let $z_0 \in \widehat{D}$ be an accumulation point of $\{f = g\}$. Then by continuity also $f(z_0) = g(z_0)$. We apply the classical identity theorem to one of the following four holomorphic functions for a suitably small $r > 0$:

(i) $f, g : B_r(z_0) \to \mathbb{C}$ when $z_0, f(z_0) \in \mathbb{C}$;
(ii) $1/f, 1/g : B_r(0) \to \mathbb{C}$ when $z_0 \in \mathbb{C}$ and $f(z_0) = \infty$;
(iii) $B_r(0) \ni z \mapsto f(\frac{1}{z}), g(\frac{1}{z})$ if $z_0 = \infty$ and $f(z_0) \in \mathbb{C}$;
(iv) $B_r(0) \ni z \mapsto \frac{1}{f(\frac{1}{z})}, \frac{1}{g(\frac{1}{z})}$ if $z_0 = f(z_0) = \infty$.

In all four cases we deduce from the classical identity theorem that $f = g$ in a neighborhood of z_0. It thus remains to show that S is also closed. Then connectedness of $\widehat{D}$ yields that $S = \widehat{D}$. Consider a point $s_0 \in \widehat{D}$ such that there exists a sequence $\{s_n\}_{n\in\mathbb{N}} \subset S$ with $s_n \to s_0$. If $s_0 \notin S$, then s_0 is also an accumulation point of $\{f = g\}$ and as above we can prove that $s_0 \in S$, which yields a contradiction. This concludes the proof. □

One can show that holomorphic functions $f : \widehat{U} \to \widehat{\mathbb{C}}$ which are not constantly ∞ can be identified with meromorphic functions since by the previous theorem the set $f^{-1}(\infty)$ is discrete in $\widehat{U} \setminus \infty$. We will show that holomorphic functions $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are exactly rational functions. To this end, we prove several auxiliary results with a similar flavor.

Lemma 7.4. *Let $f : \widehat{\mathbb{C}} \to \mathbb{C}$ be holomorphic. Then f is constant.*

Proof. Since $\widehat{\mathbb{C}}$ is compact (see exercise H 11.1), it follows that its image is also compact. Hence $f(\mathbb{C}) \subset f(\widehat{\mathbb{C}}) \subset \mathbb{C}$ is bounded, so that by Liouville's theorem $f|_{\mathbb{C}}$ is constant. By continuity of f at ∞ we conclude that f is constant on $\widehat{\mathbb{C}}$. □

When $P : \mathbb{C} \to \mathbb{C}$ is a non-constant polynomial, then one can show that $P(\infty) := \infty$ gives a holomorphic extension $P : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Our next result states that polynomials are the only class of functions for which such an extension works.

Lemma 7.5. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be holomorphic and such that $f(z) \in \mathbb{C}$ for all $z \in \mathbb{C}$. Then $f|_{\mathbb{C}}$ is a polynomial.*

Proof. See exercise H 11.3. □

The next theorem provides a complete characterization of holomorphic functions $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

Theorem 7.6. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be holomorphic. Then there exist two polynomials $P, Q : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that*

$$f(z) = \frac{P(z)}{Q(z)} \qquad \forall z \in \mathbb{C} \setminus f^{-1}(\infty).$$

Proof. See exercise H 11.4. □

Remark 7.7. Without loss of generality we can assume that P and Q have no common zeros. The statement of the above theorem then holds on $\widehat{\mathbb{C}}$ in the sense that (in general) the fraction ∞/∞ at $z = \infty$ has to be interpreted depending on the degree (and possibly the leading coefficient) of P and Q. In that sense one can also show that every rational function is holomorphic on $\widehat{\mathbb{C}}$.

With the above theorem we can easily identify the biholomorphic functions from $\widehat{\mathbb{C}}$ to itself.

Corollary 7.8. *A function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is biholomorphic if and only if f is a so-called Möbius transformation, i.e., there exist $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ and*

$$f(z) = \frac{az + b}{cz + d}$$

with the convention that $f(\infty) = a/c \in \widehat{\mathbb{C}}$ and $f(-d/c) = \infty$.

Proof. First note that when f is of the above type then f is rational, hence holomorphic on $\widehat{\mathbb{C}}$. It has its only pole in $z = -d/c$ (note that the condition $ad - bc \neq 0$ rules out cancellations in the singularity). A direct calculation shows that its inverse is given by

$$f^{-1}(z) = \begin{cases} \frac{dz-b}{-cz+a} & \text{if } z \in \mathbb{C} \setminus \{a/c\}, \\ \infty & \text{if } z = a/c \in \widehat{\mathbb{C}}, \\ -\frac{d}{c} & \text{if } z = \infty. \end{cases} .$$

This is again a rational function, so that it is holomorphic on $\widehat{\mathbb{C}}$. Thus f is biholomorphic.

Now we prove the converse statement. If $f(\infty) = \infty$, then $f(\mathbb{C}) \subset \mathbb{C}$ and by exercise H 10.4 b) we know that f is affine which yields $a \neq 0$, $b \in \mathbb{C}$, $c = 0$ and $d = 1$. Hence assume without loss of generality that $f(\infty) \in \mathbb{C}$. Composing f with the Möbius transformation

$$\varphi(z) = \frac{1}{z - f(\infty)}$$

we obtain a biholomorphic function $\tilde{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\tilde{f}(\infty) = \infty$. Hence again exercise H 10.4 b) implies that $\tilde{f}$ is affine, so that there exists $a, b \in \mathbb{C}$ with $a \neq 0$ such that $\tilde{f}(z) = az + b$ for all $z \in \mathbb{C}$. Hence by the first part of the proof

$$f(z) = (\varphi^{-1} \circ \tilde{f})(z) = \frac{-f(\infty)(az+b) - 1}{-(az+b)} = \frac{-f(\infty)az + (-1 - f(\infty)b)}{-az - b}$$

is a Möbius transform since $f(\infty)ab - a(1 + f(\infty)b) = -a \neq 0$ (from a more abstract point of view, we used that the Möbius transforms form a group with respect to the composition of functions). □

We stop here with the short introduction on holomorphic functions on the Riemann sphere. Further details should be studied from the more general viewpoint of Riemann surfaces.

8. An introduction to complex analysis in several variables

In the final chapter of the course we briefly discuss functions $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}^n$ is open ($n \geq 2$). This introduction is by no means complete and we will omit several proofs.

First let us define what we mean by holomorphic functions in higher dimensions. In what follows we let $\|\cdot\|$ be any norm on $\mathbb{C}^n$ (recall that all norms on finite dimensional spaces are equivalent).

Definition 8.1. Let $U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$. Then f is called complex-differentiable in $a \in U$ if there exists a $\underline{\mathbb{C}\text{-linear}}$ map $Df(a) : \mathbb{C}^n \to \mathbb{C}$ such that

$$\lim_{\substack{h \to 0 \\ h \neq 0}} \frac{|f(a+h) - f(a) - Df(a)h|}{\|h\|} = 0.$$

f is called holomorphic on U if f is complex differentiable in every point $a \in U$. A function $f : U \to \mathbb{C}^m$ is called holomorphic if each component is holomorphic.

Remark 8.2. Similar to the theory of one complex variable there are several equivalent definitions of holomorphic functions $f : U \to \mathbb{C}$:

(i) for each fixed $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$ the function $z \mapsto f(z_1, \ldots, z_{j-1}, z, z_{j+1}, \ldots z_n)$ is holomorphic on the open set

$$U(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) := \{z \in \mathbb{C} : (z_1, \ldots, z_{j-1}, z, z_{j+1}, \ldots, z_n) \in U\}.$$

(ii) f is C^1 in each complex variable separately and satisfies

$$\frac{\partial}{\partial \overline{z}_j} f \equiv 0,$$

where $\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}\right)$; cf. exercise H 12.1.

(iii) for each $a \in U$ there exists $r > 0$ such that on $B_r(a)$ the function f can be written as an absolutely convergent power series
$$f(z) = \sum_{\alpha} c_\alpha (z-a)^\alpha,$$
where $\alpha \in (\mathbb{N}_0)^n$ stands for a multi-index.

(iv) f is continuous in each complex variable separately and locally bounded. Moreover, for any $w \in U$ there exists $r > 0$ such that $\overline{D_r(w)} \subset U$ and for all $z \in D_r(w)$ it holds that
$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_n - w_n| = r} \cdots \int_{|\zeta_1 - w_1| = r} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} \,\mathrm{d}\xi_1 \ldots \mathrm{d}\zeta_n,$$
where $D_r(w) := \{z \in \mathbb{C}^n : \, |z_i - w_i| < r \;\forall i\}$ is the so-called polydisc.

One can show that those four conditions are all equivalent to Definition 8.1. There is however a subtle point in this statement. While Definition 8.1, (iii) and (iv) imply a local boundedness (or even continuity), this is not clear from (i) and (ii). It is indeed a deep result due to Hartogs that separately holomorphic functions are continuous (and smooth) (cf. Theorem 8.19 or [2, Section 2.4] for a proof with all details).

As in the multidimensional real-variable case one can show that the differential $Df(z_0)$ is unique, linear in f and that the chain rule holds.

Next we will introduce a technique that allows to transfer some results on holomorphic functions in one complex-variable to the several variables case. This is the so-called method of slicing, which is also used in the calculus of variations.

Lemma 8.3. *Let $U \subset \mathbb{C}^n$ be open and $a \in U$. For $\xi \in \mathbb{C}^n$ define the set $U_{a,\xi}$ by*
$$U_{a,\xi} = \{t \in \mathbb{C} : \, a + t\xi \in U\}.$$
Given a holomorphic function $f : U \to \mathbb{C}$ we define $f_{a,\xi} : U_{a,\xi} \to \mathbb{C}$ by
$$f_{a,\xi}(t) = f(a + t\xi).$$
Then $U_{a,\xi} \subset \mathbb{C}$ is open with $0 \in U_{a,\xi}$ and $f_{a,\xi}$ is holomorphic on $U_{a,\xi}$.

Proof. Clearly $a \in U$ implies $0 \in U_{a,\xi}$. Note that $U_{a,\xi}$ is open as the preimage of the open set U under the continuous map $t \mapsto a + t\xi$. By the chain rule $f_{a,\xi}$ is holomorphic on $U_{a,\xi}$. □

Corollary 8.4. *We have the analogues of the following results from the one-dimensional theory:*

1. **Liouville's theorem:** *Every bounded entire function $f : \mathbb{C}^n \to \mathbb{C}$ is constant.*
2. **Identity theorem:** *Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}$ be holomorphic. If $f|_{B_r(a)} \equiv 0$ for some $a \in D$ and $r > 0$, then $f \equiv 0$.*
3. **Open mapping theorem:** *Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}$ be non-constant and holomorphic. Then $f(D)$ is again a domain.*
4. **Maximum principle:** *Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}$ be holomorphic. If $|f|$ attains its maximum on D then f is constant.*

Proof. See exercise H 12.2. □

We saw that slices of holomorphic functions $f : U \subset \mathbb{C}^n \to \mathbb{C}$ allow to transfer some results from the one-dimensional theory to the several variables case. By a similar consideration one can prove a suitable higher-dimensional version of Cauchy's integral formula (see also Remark 8.2(iv)) which implies also several analogues of the one-dimensional theory (the proofs are almost identical).

In what follows, given a vector $r = (r_1, \ldots, r_n) \in (0, +\infty)^n$ and $a \in \mathbb{C}^n$ we define the polydisc $D_r^n(a)$ by
$$D_r^n(a) := \{z \in \mathbb{C}^n : \, |z_i - a_i| < r_i\}.$$
Then we have the following result.

Theorem 8.5 (Cauchy's integral formula for polydiscs)**.** *Let $U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$ be holomorphic. Let $a \in U$ and $r \in (0,+\infty)^n$ be such that $\overline{D_r^n(a)} \subset U$. Then for all $z \in D_r^n(a)$ it holds that*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_n - a_n| = r_n} \cdots \int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta)}{\prod_{i=1}^n (\zeta_i - z_i)} \,\mathrm{d}\zeta.$$

Proof. We prove the statement by induction on n. For $n = 1$ the claim coincides with Theorem 0.2, so there is nothing to prove. Next assume that $n \geq 2$. Given $z \in D_r^n(a)$ it follows that $z_n \in B_{r_n}(a_n)$ and $(z_1, \dots, z_{n-1}) \in D^{n-1}_{(r_1,\dots,r_{n-1})}(a_1, \dots, a_{n-1})$, whose closure is contained in the open set $U_{n-1} = \{z \in \mathbb{C}^{n-1} : (z, z_n) \in U\}$. Hence by the induction hypothesis we can write

$$f(z) = \frac{1}{(2\pi i)^{n-1}} \int_{|\zeta_{n-1} - a_{n-1}| = r_{n-1}} \cdots \int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta_1, \dots, \zeta_{n-1}, z_n)}{\prod_{i=1}^{n-1} (\zeta_i - z_i)} \,\mathrm{d}\zeta_1 \dots \mathrm{d}\zeta_{n-1}.$$

Note that for fixed $\zeta_1, \dots, \zeta_{n-1}$ in the domain of integration the function $z \mapsto f(\zeta_1, \dots, \zeta_{n-1}, z)$ is holomorphic on $B_{r_n}(a_n)$ and $\overline{B_{r_n}(a_n)} \subset U' := \{z \in \mathbb{C} : (\zeta_1, \dots, \zeta_{n-1}, z) \in U\}$. Hence applying again the one-dimensional result the claim follows from Fubini's theorem. □

Similar to the case of one complex variable Cauchy's integral formula has several consequences that we list here without proof below. The detailed arguments can be bound for instance in [3].

Corollary 8.6 (Higher dimensional consequences of the Cauchy integral formula)**.** *Let $U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$ be holomorphic. Then*

(i) $f \in C^\infty(U)$ and all derivatives are holomorphic. Moreover, in the situation of Theorem 8.5, for every multi-index $\alpha \in (\mathbb{N}_0)^n$ it holds that

$$D^\alpha f(z) = \frac{\alpha!}{(2\pi i)^n} \int_{|\zeta_n - a_n| = r_n} \cdots \int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta)}{(\zeta - z)^{\alpha + \overline{1}}} \,\mathrm{d}\zeta,$$

where $\overline{1} = (1, \dots, 1) \in \mathbb{N}_0^n$. In particular,

$$|D^\alpha f(a)| \leq \frac{\alpha!}{r^\alpha} \sup_{z \in D_r^n(a)} |f(z)|.$$

(ii) f is analytic, that is for every $w \in U$ there exists an open neighborhood V of w such that on V we have

$$f(z) = \sum_{\alpha \in (\mathbb{N}_0)^n} \frac{D^\alpha f(w)}{\alpha!} (z - w)^\alpha.$$

Moreover, the series converges uniformly on every polydisc $D_r^n(w)$ such that $\overline{D_r^n(w)} \subset U$.

Remark 8.7 (Montel's theorem)**.** Using the bound of Corollary 8.6 (i) one can prove that Montel's theorem (in the version of Chapter 1) also holds in the several variables case. Indeed, the bound implies that a locally uniformly bounded sequence $f_n : U \subset \mathbb{C}^n \to \mathbb{C}$ is locally equicontinuous. Then the rest of the proof of Montel's theorem remains unchanged. Also the local uniform limit of holomorphic functions is still holomorphic. This can be shown using the local uniform convergence on slices and the fact that a function is holomorphic if and only if it is holomorphic in each variable (here we rely on Hartogs' theorem in a simpler setting, because we know a priori that the limit is continuous)

Until now we saw that several properties still hold in the multi-dimensional setting. Next we point out some significant differences.

Remark 8.8 (Some of the differences to the one-variable setting)**.** Let $n \geq 2$.

- We will prove below that in $\mathbb{C}^n$ holomorphic functions can only have removable isolated singularities. Moreover, there cannot be isolated zeros.
- The above will be a consequence of an extension result which in a more general form reads as follows: let $U \subset \mathbb{C}^n$ be open and let $K \subset U$ be a compact set and assume that $U \setminus K$ is connected. If $f : U \setminus K$ is holomorphic then f can be extended to a holomorphic function on U (see [5, Theorem 5.4.4]).

- We will also prove that the sets $B_1(0)$ and $D_1^n(0)$ are not biholomorphically equivalent which rules out a Riemann mapping theorem.

Let us formulate the announced extension result in a special situation.

Theorem 8.9 (Special case of Hartogs' extension theorem)**.** *Let $D \subset \mathbb{C}^{n-1}$ be a domain and $A(r,R) := \{z \in \mathbb{C} : r < |z| < R\}$ with $0 \le r < R \le +\infty$. Let $f : D \times A(r,R) \to \mathbb{C}$ be holomorphic. Assume that there exists $a \in D$ and $\varepsilon > 0$ such that f can be extended holomorphically to $B_\varepsilon(a) \times B_R(0)$. Then f can be extended holomorphically to $D \times B_R(0)$.*

Proof. Denote the points in $D \times \mathbb{C}$ by (z', z_n). Given $r < \rho < R$ we define the function

$$f_\rho(z', z_n) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(z',\zeta)}{\zeta - z_n}\,\mathrm{d}\zeta.$$

This function is continuous on $D \times B_\rho(0)$ and separately holomorphic in each variable. By Hartogs' theorem (cf. Remark 8.2 or Theorem 8.19, but again here the continuity is guaranteed a priori) it is therefore holomorphic on $D \times B_\rho(0)$. By assumption there exists $\varepsilon > 0$ such that f can be extended holomorphically to the set $B_\varepsilon(a) \times B_R(0)$. By the one-dimensional Cauchy-integral formula it holds that $f = f_\rho$ on $B_\varepsilon(a) \times B_\rho(0)$. Next note that the set $D \times A(r,\rho)$ is a domain in $\mathbb{C}^n$, so that by the identity theorem we deduce that $f = f_\rho$ on $D \times A(r,\rho)$. Then the function

$$F(z', z_n) = \begin{cases} f(z', z_n) & \text{if } (z', z_n) \in D \times A(r,R), \\ f_\rho(z', z_n) & \text{if } (z', z_n) \in D \times B_\rho(0) \end{cases}$$

is well-defined, holomorphic and extends f. □

Corollary 8.10. *Let $n \ge 2$.*

(i) If $f : U \setminus \{a\} \to \mathbb{C}$ is holomorphic, then f can be extended to a holomorphic function $f : U \to \mathbb{C}$.
(ii) If $K \subset \mathbb{C}^n$ is compact and such that $\mathbb{C} \setminus K$ is connected, then every holomorphic function $f : \mathbb{C}^n \setminus K \to \mathbb{C}$ can be extended to an entire function.
(iii) If $f : U \to \mathbb{C}$ is holomorphic, then f cannot have an isolated zero.
(iv) If $f : \mathbb{C}^n \to \mathbb{C}$ is entire, then $\{f = 0\}$ is either empty or unbounded.

Proof. See exercise H 13.3. □

The next result rules out a Riemann mapping theorem in $\mathbb{C}^n$ for $n \ge 2$. In the proof we will use the following auxiliary lemma which cannot be deduced from an open mapping theorem.

Lemma 8.11. *Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}^m$ be holomorphic. If $\|f\|_2$ is constant, then f is constant. Here $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{C}^n$.*

Proof. Let us apply the differential operator $\frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$ to the equality $\|f(z)\|_2^2 = c$. By the product rule we deduce that

$$0 = \sum_{k=1}^m \frac{\partial}{\partial \overline{z_j}}(f_k(z)\overline{f_k(z)}) = \sum_{k=1}^m \frac{\partial f_k(z)}{\partial \overline{z_j}}\overline{f_k(z)} + f_k(z)\frac{\partial \overline{f_k}(z)}{\partial \overline{z_j}} = \sum_{k=1}^m f_k(z)\frac{\partial \overline{f_k}(z)}{\partial \overline{z_j}},$$

where we used that $\frac{\partial f}{\partial \overline{z_j}} = 0$ for every holomorphic function (cf. exercise H 13.1.). Now consider the differential operator $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$. Note that by definition

$$\frac{\partial \overline{f}}{\partial \overline{z_j}} = \overline{\left(\frac{\partial f}{\partial z_j}\right)}, \qquad \frac{\partial \overline{f}}{\partial z_j} = \overline{\left(\frac{\partial f}{\partial \overline{z_j}}\right)}.$$

It is a well-known fact that due to the Cauchy-Riemann equations, for holomorphic functions $\frac{\partial}{\partial z_j}$ agrees with the complex partial derivative. Hence we conclude that

$$0 = \sum_{k=1}^m \frac{\partial}{\partial z_j}\left(f_k(z)\frac{\partial \overline{f_k}(z)}{\partial \overline{z_j}}\right) = \sum_{k=1}^m \left|\frac{\partial f_k(z)}{\partial z_j}\right|^2 + f_k(z)\overline{\left(\frac{\partial^2 f}{\partial \overline{z_j}\partial z_j}\right)} = \sum_{k=1}^m \left|\frac{\partial f_k(z)}{\partial z_j}\right|^2,$$

where we used that the partial derivatives of f are still holomorphic. Since D is connected the above implies that all f_k are constant. Hence f is constant. □

Finally we will prove that in several complex variables there cannot hold a Riemann mapping theorem. Indeed, we have the following result.

Theorem 8.12 (Failure of the Riemann mapping theorem in higher dimensions)**.** *Let $n \geq 2$. Then there exists no biholomorphic map $f : D_1^n(0) \to B_1(0)$, where the ball $B_1(0)$ is defined with respect to the Euclidean metric.*

Remark 8.13. Since both $D_1^n(0)$ and $B_1(0)$ are convex, they are simply connected. Hence the above result indeed shows that the Riemann mapping theorem cannot hold in higher dimensions. In [5, Exercise 3.2.3] you find an example of a bijective function $\varphi : D_1^2(0) \to B_1(0)$ such that φ and φ^{-1} are real-analytic.

Proof of Theorem 8.12. Assume that there exists a biholomorphic function $f : D_1^n(0) \to B_1(0)$. For fixed $w \in D_1^1(0) \subset \mathbb{C}$ define the map

$$F_w : D_1^{n-1}(0) \to \mathbb{C}^n, \quad z' \mapsto \frac{\partial f}{\partial z_n}(z', w).$$

We will prove that F_w can be extended continuously to $\partial D_1^{n-1}(0)$ by 0. To this end, take a sequence $\{z_j'\}_j \subset D_1^{n-1}(0)$ such that $\lim_j z_j' \in \partial D_1^{n-1}(0)$ and define the sequence $f_j : D_1^1(0) \to B_1(0)$ by $f_j(w) = f(z_j', w)$. By Montel's theorem there exists a subsequence f_j (not relabeled) such that $f_j \to g$ locally uniformly on $D_1^1(0)$ for some holomorphic function $g : D_1^1(0) \to \overline{B_1(0)}$. By construction, for every $w \in D_1^1(0)$ the sequence $\{(z_j', w)\}_j$ converges to a point $z_w \in \partial D_1^n(0)$. We claim that $g(w) \in \partial B_1(0)$. Indeed, otherwise the continuity of f^{-1} on $B_1(0)$ implies that

$$\partial D_1^n(0) \ni \lim_j (z_j', w) = \lim_j f^{-1}(f(z_j', w)) = f^{-1}(g(w)) \in D_1^n(0),$$

which gives a contradiction since $D_1^n(0)$ is open. Hence $g(D_1^1(0)) \subset \partial B_1(0)$. By the previous lemma g is constant. Hence Theorem 1.5 implies that

$$0 = g'(w) = \lim_j f_j'(w) = \lim_j F_w(z_j').$$

Since the sequence was arbitrary (and the result is independent of the subsequence) it follows that F_w can be extended continuously to $\partial D_1^{n-1}(0)$ via 0. Applying the maximum principle to each coordinate of F_w we deduce that $F_w \equiv 0$. By definition of F_w we conclude that $\det(Df(z', w)) = 0$. However, by the chain rule

$$Id = Df^{-1}(f(z', w))Df(z', w),$$

so that $Df(z', w)$ has a trivial nullspace. This yields a contradiction. □

As a final result, we will prove Hartogs' theorem on separate holomorphy. For the proof we will need some results on subharmonic functions and the following definition.

Definition 8.14. Let X be a metric space. A function $u : X \to \mathbb{R} \cup \{\pm\infty\}$ is called lower semicontinuous if for every $x_0 \in X$ and every sequence $x_n \to x_0$ it holds that

$$u(x_0) \leq \liminf_{n \to +\infty} u(x_n).$$

It is called upper semicontinuous if for every $x_0 \in X$ and every sequence $x_n \to x_0$ it holds that

$$u(x_0) \geq \limsup_{n \to +\infty} u(x_n).$$

We will use the following elementary properties of lower semicontinuous functions.

Lemma 8.15. *Let X be a metric space and I be a set of indices. If for all $i \in I$ the function $u_i : X \to \mathbb{R} \cup \{\pm\infty\}$ is lower semicontinuous, then the function $\bar{u}(x) := \sup_{i \in I} u_i(x)$ is lower semicontinuous. Moreover, a function $u : X \to \mathbb{R} \cup \{\pm\infty\}$ is lower semicontinuous if and only if the set $\{x \in X : u(x) \leq t\}$ is closed for all $t \in \mathbb{R}$[1].*

[1] In general topological spaces the closedness of sublevel sets can be taken as the definition of lower semicontinuity.

Proof. Let $x \in X$ and consider a sequence $x_n \to x$. Then

$$\bar{u}(x) = \sup_{i\in I} u_i(x) \leq \sup_{i\in I} \liminf_{n\to+\infty} \underbrace{u_i(x_n)}_{\leq \bar{u}(x_n)} \leq \sup_{i\in I} \liminf_{n\to+\infty} \bar{u}(x_n) = \liminf_{n\to+\infty} \bar{u}(x_n).$$

Hence $\bar{u}$ is lower semicontinuous. To prove the second assertion, assume first that u is lower semicontinuous and that $x_n \in \{x \in X : u(x) \leq t\}$ for all $n \in \mathbb{N}$ and that $x_n \to x$ for some $x \in X$. Then by the lower semicontinuity of f we have that

$$u(x) \leq \liminf_{n\to+\infty} \underbrace{u(x_n)}_{\leq t} \leq t.$$

Hence $x \in \{x \in X : u(x) \leq t\}$. Next, assume that the latter set is closed for all $t \in \mathbb{R}$. Fix $x_0 \in X$ and a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \to x$. If $\liminf_{n\to+\infty} u(x_n) = +\infty$, then there is nothing to prove. Hence assume that the liminf is not $+\infty$. Passing to a subsequence realizing the liminf we can assume that the liminf is actually a limit that we denote by $u_0 \in \mathbb{R} \cup \{-\infty\}$. Fix any $t > u_0$. Then for all n large enough we have that $x_n \in \{x \in X : u(x) \leq t\}$. Since the set is closed, it follows that $u(x_0) \leq t$. This holds for any $t > u_0$, so that actually $u(x_0) \leq u_0$. This proves the lower semicontinuity u. □

We next state the definition of subharmonic functions.

Definition 8.16. Let $U \subset \mathbb{C}$ be open and $u : U \to \mathbb{R} \cup \{-\infty\}$. We say that u is subharmonic in U if u is upper semicontinuous and for all $z_0 \in U$ there exists $\delta > 0$ such that for all $0 < r < \delta$[2] it holds that

$$u(z_0) \leq \frac{1}{2\pi r} \int_{\partial B_r(z_0)} u(z)\,\mathrm{d}z. \tag{8}$$

Remark 8.17. An important class of a subharmonic functions is given by the following example: if $g : U \to \mathbb{C}$ is holomorphic, then

$$u(z) = \begin{cases} \log(|g(z)|) & \text{if } g(z) \neq 0, \\ -\infty & \text{if } g(z) = 0, \end{cases}$$

is subharmonic. Indeed, such a function is clearly upper semicontinuous. Moreover, if z_0 is such that $g(z_0) = 0$, then (8) obviously holds. If $g(z_0) \neq 0$, then locally we can find a holomorphic logarithm $\log(g)$ and $\log(|g|) = \mathrm{Re}(\log(g))$.[3] Thus (8) holds with equality using the mean-value for holomorphic functions, which is a consequence of Cauchy's integral formula.

We will use the following compactness result for sequences of subharmonic functions without proof.

Lemma 8.18. *Let $u_n : U \to \mathbb{R} \cup \{-\infty\}$ be a sequence of subharmonic functions such that there exists $M > 0$ and $c \in \mathbb{R}$ with*

$$\begin{aligned} u_n(z) &\leq M \quad \text{for all } n \in \mathbb{N},\ z \in U, \\ \limsup_{n\to+\infty} u_n(z) &\leq c \quad \text{for all } z \in U. \end{aligned}$$

Then for every compact set $K \subset U$ and all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ it holds that

$$\sup_{z\in K} u_n(z) \leq c + \varepsilon.$$

Theorem 8.19 (Hartogs' theorem on holomorphy)**.** *Let $f : U \to \mathbb{C}$ be holomorphic separately in each variable. Then f is holomorphic in the sense of Definition 8.1.*

[2] Using arguments from the theory of partial differential equations one can show that this local definition of subharmonic functions is equivalent to require the mean-value inequality for all closed balls contained in U.

[3] It holds that

$$|g| = |e^{\log(g)}| = |e^{\mathrm{Re}(\log(g))+i\mathrm{Im}(\log(g))}| = |e^{\mathrm{Re}(\log(g))}e^{i\mathrm{Im}(\log(g))}| = e^{\mathrm{Re}(\log(g))},$$

so that $\log(|g|) = \mathrm{Re}(\log(g))$.

Proof. We shall show that f can locally be written as an absolutely converging power series, that is, for all $z_0 \in U$ there exists $r > 0$ such that

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha (z - z_0)^\alpha \qquad \text{for all } z \in D_r^n(z_0).$$

The claim then follows by general properties of converging power series (one shows that all partial derivatives exist and are continuous. Then one can prove the differentiability as on $\mathbb{R}^n$).

We split it into several steps. Without loss of generality we can assume that $z_0 = 0$ and let us choose $r > 0$ such that $\overline{D_{2r}^n(0)} \subset U$.

Step 1: Assuming local boundedness
Here we assume that f is bounded on $\overline{D_r^n(0)}$. Iterating Cauchy's integral formula, the separate holomorphy of f implies that for all $z \in D_r^n(0)$ we have

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\xi_1|=r} \cdots \int_{|\xi_n|=r} \frac{f(\xi)}{\prod_{j=1}^n (\xi_j - z_j)} \, \mathrm{d}\xi_n \ldots \mathrm{d}\xi_1.$$

For $|z_j| < |\xi_j|$, using the geometric series formula we have that

$$\frac{1}{\xi_j - z_j} = \sum_{n_j=0}^{\infty} \frac{z_j^{n_j}}{\xi_j^{n_j+1}}$$

and the sum converges absolutely and uniformly with respect to $|\xi_j| = r$ and $|z_j| \leq r_0 < r$. Using the boundedness of f, the local uniform convergence of the geometric series allows us to switch sums and integrals to obtain

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha, \qquad c_\alpha = \frac{1}{(2\pi i)^n} \int_{|\xi_1|=r} \cdots \int_{|\xi_n|=r} \frac{f(\xi)}{\xi^{\alpha+(1,\ldots,1)}} \, \mathrm{d}\xi_n \ldots \mathrm{d}\xi_1$$

and the series converges absolutely and locally uniformly on $D_r^n(0)$. This is the claimed power series representation and we conclude that f is holomorphic in the sense of Definition 8.1.

We now start an induction proof on the dimension n. For $n = 1$ the statement is clear. Next, assuming that f is jointly holomorphic in the first $n-1$-variables and holomorphic in the last variable and we will show that f is locally bounded, hence jointly holomorphic in n variables.

Step 2: Local boundedness in a smaller polydisc via the Baire category theorem
We claim that there exist closed discs $E_j \subset \overline{B_r(0)} \subset \mathbb{C}$ with non-empty interior and $E_n = \overline{B_r(0)}$ such that f is bounded on $E_1 \times \ldots \times E_n$. Note, however, that in general $0 \notin E_j$, so this step is not sufficient to prove complex-differentiability in the origin. Given $N \in \mathbb{N}$ we define the sets

$$\Omega_N := \left\{ z' \in \prod_{j=1}^{n-1} \overline{B_r(0)} : \sup_{z_n \in \overline{B_r(0)}} |f(z', z_n)| \leq N \right\}.$$

Using the induction hypothesis, we know that the function $z' \mapsto f(z', z_n)$ is in particular continuous, so that by Lemma 8.15 the function $z' \mapsto \sup_{z_n \in \overline{B_r(0)}} |f(z', z_n)|$ is lower semicontinuous. Hence, again by Lemma 8.15, the set Ω_N is a closed set. Moreover, for any fixed $z' \in \prod_{j=1}^{n-1} \overline{B_r(0)}$ the function $z_n \to f(z', z_n)$ is holomorphic on $B_{2r}(0)$ and thus bounded on $\overline{B_r(0)}$. It follows that

$$\bigcup_{N \in \mathbb{N}} \Omega_N = \prod_{j=1}^{n-1} \overline{B_r(0)}.$$

The set on the right-hand side has non-empty interior, so that by the Baire category theorem there exists $N \in \mathbb{N}$ such that Ω_N has non-empty interior. In particular, this set Ω_N contains a closed polydisc with non-empty interior. The definition of the set Ω_N yields the claim of Step 2.

Step 3: From boundedness on smaller to boundedness on larger polydiscs via subharmonic functions
We show that if $f : D_r^n(z_0) \to \mathbb{C}$ is separately holomorphic in z' and in z_n and bounded in a smaller

polydisc $D^n_{(r',\dots,r',r)}(z_0)$ with $r' < r$, then it is holomorphic on $D^n_r(z_0)$. Note that the center of the two polydiscs is the same in contrast to what we obtained in Step 2. To reduce notation, we assume for simplicity again that $z_0 = 0$. By assumption, for $(z', z_n) \in D^n_r(0)$ we can write

$$f(z', z_n) = \sum_{\alpha\in\mathbb{N}_0^{n-1}} c_\alpha(z_n)(z')^\alpha, \tag{9}$$

the series converges absolutely and locally uniformly with respect to z' and the coefficients are given by

$$c_\alpha(z_n) = \frac{\partial^\alpha}{\partial(z')^\alpha}\frac{f(0, z_n)}{\alpha!}.$$

Due to the holomorphy in z' we can use Cauchy's integral formula for the derivative (cf. Corollary 8.6) to obtain that

$$c_\alpha(z_n) = \frac{1}{(2\pi i)^n}\int_{|\xi_1|=\frac{r'}{2}}\cdots\int_{|\xi_{n-1}|=\frac{r'}{2}} \frac{f(\xi_1,\dots,\xi_{n-1},z_n)}{\xi_1^{\alpha_1+1}\dots\xi_{n-1}^{\alpha_{n-1}+1}}\,\mathrm{d}\xi_{n-1}\dots\,\mathrm{d}\xi_1. \tag{10}$$

Since f is bounded and thus holomorphic on $D^n_{(r',\dots,r',r)}(z_0)$, one can use a result from measure theory about the differentiability of integrals with respect to a parameter to show that c_α is complex-differentiable on $B_r(0)\subset\mathbb{C}$. Hence $v_\alpha(z_n) = \frac{1}{|\alpha|}\log(|c_\alpha(z_n)|)$ defines a family of subharmonic functions on $B_r(0)$.

We next verify the assumption of Lemma 8.18, implicitly numbering the countably many multi-indices $\alpha\in\mathbb{N}_0^{n-1}$, so that v_α can be seen as a sequence. First, note that since the sum in (9) converges absolutely and locally uniformly with respect to z' it follows that for any $0 < r_2 < r$

$$\lim_{|\alpha|\to+\infty}|c_\alpha(z_n)|r_2^{|\alpha|} = 0 \quad \text{ for all } z_n\in B_r(0),$$

which implies that pointwise

$$\limsup_{|\alpha|\to+\infty} v_\alpha(z_n) \le -\log(r_2).$$

Moreover, from (10) we deduce from the standard estimate for contour integrals that

$$|c_\alpha(z_n)| \le \frac{\sup_{z'\in D^{n-1}_{r'/2}(0)}|f(z',z_n)|}{\left(\frac{r'}{2}\right)^{|\alpha|}} \le \frac{B}{\left(\frac{r'}{2}\right)^{|\alpha|}},$$

where B is a bound for $|f|$ on the smaller polydisc $D^n_{r',\dots,r',r}(0)$. Taking the logarithm, we find that for all $z_n\in B_r(0)$ it holds that

$$v_\alpha(z_n) \le \frac{1}{|\alpha|}\log(B) - \log(\tfrac{r'}{2}) \le \log(B) - \log(\tfrac{r'}{2}).$$

Hence v_α is uniformly bounded from above and we can apply Lemma 8.18 to deduce that for all $0 < r_1 < r_2$ there exists $N\in\mathbb{N}$ such that for all $|\alpha|\ge N$ and all $z_n\in B_{r_1}(0)$ it holds that $v_\alpha(z_n)\le -\log(r_1)$, which is equivalent to

$$|c_\alpha(z_n)|r_1^{|\alpha|} \le 1 \quad \text{ for all } z_n\in B_{r_1}(0).$$

From exercise H 12.4 b) we thus infer that the series

$$f(z', z_n) = \sum_{\alpha\in\mathbb{N}_0^{n-1}} c_\alpha(z_n)(z')^\alpha$$

converges uniformly on $\overline{D^n_{r_0}(0)}$ for all $0 < r_0 < r_1$. In particular, it is bounded on this set and hence jointly holomorphic on the interior by Step 1. Since the radii $r_0 < r_1 < r_2 < r$ were arbitrary, we deduce that f is holomorphic on $D^n_r(0)$.

Step 4: Geometric conclusion

By Step 2 we find a closed polydisc $\overline{D^{n-1}_{r'}}(z_0)\times\overline{B_r(0)}$ on which f is bounded. In addition, we know that $(z_0, 0)\in D^n_r(0)$ since the closed polydisc is a subset of $\overline{D^n_r(0)}$. By Step 3 we know that f is holomorphic on $D^{n-1}_r(z_0)\times B_r(0)$ and the inclusion $z_0\in D^{n-1}_r(0)$ yields that $(0', 0)\in D^{n-1}_r(z_0)\times B_r(0)$. This concludes the proof. □

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