Week #11

Algebra V - Galois theory

Nov 29, 2024

Problem 1. *Let K be a field, and let $L = K(\alpha)$ be a field extension of K. Let $f(x)$ be the minimal polynomial of α (over K). Assume that f has degree n and prove that*

$$\Delta(f) = (-1)^{\frac{n(n-1)}{2}} N_{L/K}(f'(\alpha)),$$

where f' denotes the (formal) derivative of f.

Problem 2. *For each of the following polynomials $f(x) \in \mathbb{Q}[X]$, compute the Galois group of the extension $SF_{\mathbb{Q}}(f)/\mathbb{Q}$.*

(i) $f(x) = x^4 + x^3 + x^2 + 1$.

(ii) $f(x) = x^4 - 4x^3 + 4x^2 + 6$.

Problem 3. *For $\alpha \in \mathbb{Q}$, let $f_\alpha(x) = x^4 + x^3 + x^2 + x + \alpha$ and let G_α denote the Galois group of the extension $SF_{\mathbb{Q}}(f_\alpha)/\mathbb{Q}$. Find four integers $\alpha_1, \alpha_2, \alpha_3$ and α_4 such that the corresponding groups G_{α_i} are pairwise distinct (not isomorphic).*

Problem 4. *Let p be a prime and let $n \geq 4$. Prove that there does not exist a polynomial $f \in \mathbb{F}_p[x]$ of degree n such that $SF_{\mathbb{F}_p}(f_\alpha)/\mathbb{F}_p$ has Galois group isomorphic to A_n or S_n. What happens if $n = 3$?*

Problem 5. *Consider the polynomial $f(x) = x^5 + 20x + 16 \in \mathbb{Q}[x]$. It is known that the Galois group of the extension $SF_{\mathbb{Q}}(f_\alpha)/\mathbb{Q}$ has order at least 15. Prove that this group must be (isomorphic to) A_5.*