Weeks #9 & 10

Algebra V - Galois theory

Nov 15 and Nov 22, 2024

Problem 1. *Let K be a field containing μ_n for some n prime to char(K). Choose $a, b \in K$ and let $\alpha, \beta \in \bar{K}$ be such that $a = \alpha^n$ and $b = \beta^n$. Assuming that $[K(\alpha) : K] = [K(\beta) : K] = n$, prove that $K(\alpha) = K(\beta)$ if and only if there exist some $r \in (\mathbb{Z}/n\mathbb{Z})^\times$ and some $c \in K^\times$ such that $a = b^r c^n$.*

Problem 2. *Let $K = \mathbb{C}(x, y, z)$ be the field of rational functions in three variables, and let $L = K(\sqrt[4]{xyz}, \sqrt[4]{y^2 z}, \sqrt[4]{xz^2})$. Prove that the extension L/K is 4-Kummer of degree 32.*

Problem 3. *Let $p_1, \ldots, p_n$ be distinct primes. Prove that $[\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n}), \mathbb{Q}] = 2^n$.*

Problem 4. *Use the Artin-Schreier theorem to prove that any nontrivial torsion element of* $\mathrm{Aut}(\bar{\mathbb{Q}})$ *has order two.*

Problem 5. *Let G be a group of order n and M be a G-module. Prove that $nH^2(G, M) = \{0\}$ by proving that given a 2-cocycle $f : G \times G \to M$, nf is cohomologous to the coboundary g given by*

$$g(\sigma, \tau) = \sum_{g \in G} f(\sigma, g) + \sum_{g \in G} \sigma \cdot f(\tau, g) - \sum_{g \in G} f(\sigma\tau, g).$$

Problem 6. *Let $G = \langle \sigma \rangle$ be a cyclic group of order n and M a G-module. Define*

$$M^G := \{m \in M \ ; \ \sigma \cdot m = m\}$$

and the corresponding trace map $T : M \to M^G$ given by $m \mapsto \sum_{g \in G} g \cdot m = \sum_{i=0}^{n-1} \sigma^i \cdot m$.

(i) Given $m \in M^G$, let $f_m : G \times G \to M$ be the cochain given by $f_m(\sigma^i, \sigma^j) := \begin{cases} 0 & \text{if } i + j < n \\ m & \text{if } i + j \geq n \end{cases}$. Prove that f_m is a 2-cocycle.

(ii) Prove that the map $m \mapsto f_m$ induces an isomorphism (of groups) $M^G / \mathrm{im}(T) \simeq H^2(G, M)$.

Problem 7. *Let G and M be as in the previous exercise. Consider the map $D : M \to M$ given by $m \mapsto \sigma \cdot m - m$.*

(i) Given $m \in M^G$ with $T(m) = 0$, let $g_m : G \to M$ be the cochain given by $g_m(\sigma^i) = \sum_{j=0}^{i-1} \sigma^j \cdot m$. Prove that g_m is a 1-cocycle.

(ii) Prove that the map $m \mapsto g_m$ induces an isomorphism (of groups) $\ker(T)/\operatorname{im}(D) \simeq H^1(G, M)$.

Problem 8. *Let L/K be a finite Galois extension with Galois group G.*
Fix $f \in \operatorname{Hom}(G \times G \to L^\times)$ and define a K-algebra as follows.

- *Consider a $|G|$-dimensional vector space V_f over L with basis $\{e_g\}_{g \in G}$. That is,*

$$V_f \simeq \bigoplus_{g \in G} Le_g.$$

- *Define a product $\star : V_f \times V_f \to V_f$, using f, by*

$$ae_\sigma \star be_\tau := a\sigma(b)f(\sigma, \tau)e_{\sigma\tau} \qquad \forall \sigma, \tau \in G \text{ and } \forall a, b \in L.$$

(i) Prove that the product $\star$ defined above makes V_f into an associative algebra if and only if f is a 2-cocycle.

(ii) Prove that if further $f(\sigma, \tau) = 1$ for all $\sigma, \tau \in G$, then this algebra is isomorphic to $\mathbb{M}_n(K) \simeq \operatorname{End}_K(L)$, where $n = |G|$.

(iii) Prove that if g is another 2-cocycle that differs from f by a 2-coboundary, then $V_g \simeq V_f$ (as K-algebras).

(iv) Let now $K = \mathbb{R}$ and $L = \mathbb{C}$. Find $[f] \in H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times)$ such that $V_f \simeq \mathbb{H}$ (as $\mathbb{R}$-algebras).