Week #7

Algebra V - Galois theory

Oct 30 and Nov 1st, 2024

Problem 1. *Let n be any positive integer and let $\Phi_n(x) = \prod_{i=1}^{\phi(n)}(x - \zeta_i)$ denote the n-th cyclotomic polynomial. That is, $\Phi_n(x) \in \mathbb{C}[x]$ is a monic polynomial, and its roots are exactly the primitive n-th roots of unity in $\mathbb{C}$. Prove that $\Phi_n(x) \in \mathbb{Z}[x]$ and $\Phi_n(x)$ is irreducible over $\mathbb{Q}$.*

Problem 2. *Let n be any positive integer and let $\mathbb{Q}_n \coloneqq SF_{\mathbb{Q}}(x^n - 1)$.*

(i) Prove that $[\mathbb{Q}_n : \mathbb{Q}] = \phi(n)$ and $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times$.

(ii) If m is an odd (positive) integer, prove that $\mathbb{Q}_{2m} = \mathbb{Q}_m$.

(iii) Find n such that $\mathbb{Q}_n$ contains a subfield which is not a cyclotomic extension of $\mathbb{Q}$.

(iv) If $\alpha \in \mathbb{Q}$, prove that $\mathbb{Q}(\sqrt{\alpha}) \subset \mathbb{Q}_n$ for some n (This is the Kronecker-Weber theorem for quadratic extensions. See also exercise 3-2 in Milne's notes).

(v) Find all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}_8$, and between $\mathbb{Q}$ and $\mathbb{Q}_{12}$.

Problem 3. *Prove that $\mathbb{Q}(\cos(2\pi/n))/\mathbb{Q}$ is a Galois extension for every $n \in \mathbb{N}$.*

Problem 4. *Let $L/\mathbb{Q}$ be a finite extension. Prove that there is only a finite number of roots of unity in L.*

Problem 5. *Let $K = \mathbb{F}_p$, with p prime. Let L be the field obtained from K by adjoining all primitive q-th roots of unity for all primes $q \neq p$. Prove that L is algebraically closed.*

Problem 6. *Consider a radical extension L/K as defined in the lecture. That is, we can find a tower*

$$K = K_0 \subset K_1 \subset \ldots \subset K_n = L$$

where each K_{j+1}/K_j is obtained by extracting an m_j-th root. Let $m = \prod_{j=1}^{n} m_j$, assume that $(char(K), m_j) = 1$ for all $j = 1, \ldots, n$ and consider the field $F = K(\mu_m)$. Prove that the tower of composite fields

$$K \subset F = FK_0 \subset FK_1 \subset \ldots \subset FK_n = FL$$

is a radical tower and that each FK_{j+1}/FK_j is Galois with cyclic Galois group.

Problem 7. *Let L/K and M/K be two finite field extensions such that $L, M \subset \bar{K}$ and M/K and LM/M are solvable. Prove that L/K is solvable as well.*

Problem 8. *Let L/K and M/K be two finite Galois extensions. Prove that LM/M and $L/L \cap M$ are also Galois and that*

$$\mathrm{Gal}(LM/M) \simeq \mathrm{Gal}(L/L \cap M)$$

Problem 9. *Let L/K be a finite Galois extension with Galois group G and let $\{\sigma(\alpha)\,;\, \sigma \in G\}$ be a normal basis.*

(i) Let $H \leq G$ be any subgroup and $\alpha_H := Tr_{L/L^H}(\alpha) = \sum_{h \in H} h(\alpha)$. Prove that $L^H = K(\alpha_H)$.

(ii) Prove that if $H \trianglelefteq G$, then a normal basis for L^H/K is given by $\{\tilde{\sigma}(\alpha_H)\,;\, \tilde{\sigma} \in G/H\}$

Problem 10. *Prove that every finite extension of a finite field is solvable.*

Problem 11. *Let L/K be a finite separable extension of prime degree p. Let $\alpha \in L$ be such that $L = K(\alpha)$ and let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_p$ be the K-conjugates of α in $\bar{K}$. Prove that if $\alpha_2 \in L$, then L/K is Galois with cyclic Galois group.*

Problem 12. *Let K be a field that does not contain a primitive fourth root of unity. Let $L = K(\sqrt{\alpha})$ for some $\alpha \in K$ that is not a square. Prove that α is a sum of two squares in K if and only if L lies in a cyclic extension of K of degree four.*