Worksheet #4

Algebra V - Galois theory

October 4, 2024

Problem 1. *Prove that any field extension of degree two is normal.*

Problem 2. *Let L/K be an algebraic field extension. Prove that L/K is normal if and only if for any K-homomorphism $\sigma : L \to \bar{K}$ we have that $\sigma(L) = L$.*

Problem 3. *Let L/K be a finite Galois extension of degree $2d$, with d odd. Prove that $\mathrm{Gal}(L/K)$ has a subgroup of index d.*

Problem 4. *Let K be a field, $f \in K[x]$ an irreducible polynomial of degree four and $L = SF_K(f)$. Prove that $\mathrm{Gal}(L/K)$ cannot be isomorphic to the quaternion group of order eight.*

Problem 5. *Let L/K be a Galois extension. Let $H \leq \mathrm{Gal}(L/K)$ and $F = L^H$ the fixed field of H. Prove that $H \trianglelefteq \mathrm{Gal}(L/K)$ if and only if F/K is normal, in which case $\mathrm{Gal}(F/K) \simeq \mathrm{Gal}(L/K)/H$.*

Problem 6. *For each of the following field extensions, prove that they are Galois and describe the corresponding Galois groups. Find all subgroups and the corresponding fixed fields.*

(i) $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$

(ii) $\mathbb{Q} \subset \mathbb{Q}(\alpha)$, *where α is a primitive 5-th root of unity.*

Problem 7. *Let $K \subset L$ be finite fields and let $|K| = q$.*

(i) Prove that the extension L/K is Galois.

(ii) Prove that $\mathrm{Gal}(L/K)$ is cyclic and generated by the Frobenius automorphism $x \mapsto x^q$.

(iii) Prove that every element of K is the norm of an element of L (see worksheet 3, exercise 8).

Problem 8. *Let $K = \mathbb{F}_2$ and $L = \mathbb{F}_2(\alpha)$, where α is a root of the polynomial $x^4 + x + 1 \in K[x]$. Prove that $\mathrm{Gal}(L/K)$ contains exactly one subgroup H of order two and find $\beta \in L$ such that $L^H = \mathbb{F}_2(\beta)$.*