Worksheet #3

Algebra V - Galois theory

September 27, 2024

Problem 1. *Let $L = \mathbb{Q}(\sqrt[4]{2})$ and $F = \mathbb{Q}(\sqrt{2})$.*

(i) Determine whether or not each of the three extensions $L/\mathbb{Q}$, $F/\mathbb{Q}$ and L/F is normal. You must justify your answer.

(ii) Find a $\mathbb{Q}$-homomorphism $\varphi : F \to F$ that does not extend to a $\mathbb{Q}$-homorphism $\tilde{\varphi} : L \to L$.

Problem 2. *Let L/K be a finite field extension of degree m, and let $f \in K[x]$ be irreducible (over K) of degree n. Prove that if $gcd(m, n) = 1$, then f is also irreducible over L.*

Problem 3. *Give an example of fields $K \subset F \subset L$ such that the extensions L/F and F/K are normal, but the extension L/K is not.*

Problem 4. *Let L/K be a finite field extension and assume that $char(K)$ does not divide $[L : K]$. Prove that L is separable over K.*

Problem 5. *Let L be a field and fix $\sigma \in Aut(L)$ of infinite order. Let $K \coloneqq \{a \in L \mid \sigma(a) = a\}$ be the fixed field of σ. Prove that if L/K is algebraic, then L is normal over K.*

Problem 6. *Let $K \subset F \subset L$ be fields and prove that the following statements are true.*

(i) If L/K is a separable extension, then L/F and F/K are separable as well.

(ii) If L/F is normal and F/K is purely inseparable, then L/K is normal.

Problem 7. *Give three examples of a field extension L/K which is neither normal nor separable.*

Problem 8. *Let L/K be a finite field extension and pick $\alpha \in L$. The trace (resp. the norm) of α is defined to be the trace (resp. the determinant) of the linear map $M_\alpha : L \to L$ of K-vector spaces given by multiplication by α. Assume that $d = [L : K(\alpha)]$ and write*

$$f \coloneqq min_K(\alpha) = x^n + a_{n_1}x^{n-1} + \ldots + a_1x + a_0 \in K[x].$$

Prove that

(i) the characteristic polynomial of M_α is f^d, and

(ii) the trace (resp. the norm) of α is the number db_{n-1} (resp. b_0^d), where $b_i \coloneqq (-1)^{n-i}a_i$.