

Worksheet #1

Algebra V - Galois theory

September 13, 2024

You will need the following definitions to solve the exercises in this worksheet.

Definition 1. Let $Q \neq R$ be two distinct points “drawn” in an infinite piece of paper, which we think of as the plane $\mathbb{R}^2$. A point $P \in \mathbb{R}^2$ is **constructible from Q and R** if $P \in \{Q, R\}$ or else P lies at the intersection of a pair of lines, of a line and a circle, or of a pair of circles that you can draw using only a straightedge and a compass following the two rules below:

Rule 1 you can draw the line (using the straightedge) through any two points that are already drawn,

Rule 2 you can open the compass to span the distance $r := d(P, \tilde{P})$ between any two points P and $\tilde{P}$ that you have already drawn, place the base at a third already drawn point, say O , and draw the circle centered at O with radius r .

Definition 2. Identify the field of complex numbers $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. A complex number $z = x + iy$ is **constructible** if and only if the point $P = (x, y)$ is constructible from $Q = (0, 0) \simeq 0_{\mathbb{C}}$ and $R = (1, 0) \simeq 1_{\mathbb{C}}$ as in Definition 1.

Problem 1. Use induction to prove that every integer is constructible.

Problem 2. Use the claims below to prove that the set F of all constructible complex numbers forms a field such that $\mathbb{Q}(i) \subset F \subset \mathbb{C}$.

Claim 1 If L is a line and P is a point that have already been drawn, then we can draw the unique line containing P that is parallel to L .

Claim 2 If α and β are constructible, then so are $\alpha \cdot \beta$ and α/β (if $\beta \neq 0$).

Problem 3. Prove that the field F (from the previous problem) is closed under taking square roots.

Problem 4 (a bit harder). Prove that you cannot construct the real number $\sqrt[3]{2}$ by proving that if $\alpha \in \mathbb{R}$ is constructible, then α is algebraic over $\mathbb{Q}$ and the degree of the field extension $\mathbb{Q} \subset \mathbb{Q}(\alpha)$ is a power of two¹.

¹Recall that this means that α is a root of a non-zero polynomial with coefficients in $\mathbb{Q}$ and the minimal such polynomial has degree a power of two.