Week #13 (non-examinable)

Algebra V - Galois theory

Dec 13, 2024

In this worksheet, you will prove the following result.

Theorem 1. *Let $f(t,x) \in \mathbb{Q}(t)[x]$ be a monic polynomial (in the variable x). For $C > 0$, let*

$$N(C) \coloneqq \#\{b \in [0,C] \cap \mathbb{Z} \ ; \ f_b = f(b,x) \in \mathbb{Q}[x] \text{ has a root in } \mathbb{Q}\}$$

and suppose that $f(t,x)$ has no root in $\mathbb{Q}(T)$. Then there exists $\alpha < 1$ such that for all $C \geq 1$ we have that $N(C) \ll C^{\alpha}$.

(i) Let $u = 1/t \in \mathbb{Q}(t)$ and prove that given $f(t,x)$ as above there exists $g(x) \in \mathbb{Q}(x)$ such that $g(u)f(1/u,x) \in \mathbb{Q}[u,x]$.

(ii) Prove that, further, there exists $n \in \mathbb{N}$ and $h \in \mathbb{Q}[u,y]$ monic (in y) such that $g(u)^n f(1/u,x) = h(u, g(u)x)$.

(iii) Prove that there exists an integer $q \geq 1$ and Laurent series (around 0 and of positive convergence radius) $\psi_1, \dots, \psi_n$ such that for all $t \in \mathbb{C}$ with $|t|$ large enough we have that

$$f(t^q, x) = \prod_{i=1}^{n} (x - \psi_i(1/t)) \in \mathbb{C}[x]$$

(iv) Use the first two points to argue that it is sufficient to prove Theorem 1 for a polynomial $f(t,x) \in \mathbb{Z}[t,x]$ monic (in the variable x) and with no roots in $\mathbb{Q}(t)$.

(v) Let $\psi(t)$ be a Laurent series which is not contained in $\mathbb{C}[[t]]$ and which converges for all $|t| \geq C_0$, where C_0 is some positive real number. For $C > 0$, denote by $\tilde{N}(C)$ the cardinality of the set $\{t \in [0,C] \cap \mathbb{Z} \ ; \ |t| \geq C_0 \text{ and } \psi(1/t) \in \mathbb{Z}\}$. Prove that there exists $\beta < 1$ such that for all $C \geq 1$ we have that $\tilde{N}(C) \ll C^{\beta}$.

(vi) Conclude that Theorem 1 holds.