December 2024

Important results in problem sets

Exercise 1. (PS 1). In class we discussed representations of associative algebras. There is a similar notion of a representation of a group. Namely, if G is a group, then a representation ρ of G over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space V together with a group homomorphism

$$\rho : G \to \mathrm{GL}(V),$$

where $\mathrm{GL}(V)$ is the group of all invertible linear transformations of the vector space V.

Show that the non-isomorphic representations of a finite group G over a field $\mathbb{K}$ are in one-to-one correspondence with the non-isomorphic representations of the algebra $\mathbb{K}[G]$.

Exercise 2. (PS 2). Consider the groups D_n given by generators and relations as follows:

$$D_n = \langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2)^n = 1\rangle.$$

(a) Classify the 1-dimensional representations of D_n up to isomorphism (the answer depends on the parity of n).

(b) For $k \in \{0, \ldots, n-1\}$, consider the following maps:

$$\rho_k(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho_k(s_2) = \begin{pmatrix} 0 & \omega^{-k} \\ \omega^k & 0 \end{pmatrix}$$

where $\omega = e^{2\pi i/n}$. Find for which values of k the map ρ_k defines an irreducible representation of $\mathbb{C}[D_n]$.

(c) Find the number of non-isomorphic irreducible representations ρ_k. The answer depends on the parity of n.

Exercise 3. (PS 4). Let (V, ρ) be a representation of an associative algebra A over $\mathbb{C}$. A vector $v \in V$ is cyclic if $\rho(A)v = V$ (the vector v generates V as an A-module). A representation admitting a cyclic vector is called cyclic. Show that a representation V is cyclic if and only if it is isomorphic to the representation A/I, where A acts by left multiplication, for a proper left ideal $I \subset A$.

Exercise 4. (PS 4). Use the isomorphism $S \cong A/I$ for a maximal left ideal $I \subset A$ to show that a simple module S over $A = \mathrm{Mat}_n(\mathbb{C})$ is isomorphic to $\mathbb{C}^n$.

Exercise 5. (PS 5). Let $A = \mathrm{Mat}_n(k)$ for a field k. Prove that the algebra A is semisimple, meaning that any finite dimensional representation of A over k is isomorphic to a direct sum of irreducible representations.
Hint: Consider the basis of matrices with a single nonzero matrix element $\{E_{ij}\}$ in A. Show that for a representation V of A, we have $V = \oplus_{i=1}^n E_{ii}V$ and that for $v \in E_{11}V$, the linear span of $\{E_{11}v, E_{21}v, \ldots E_{n1}v\}$ is a subrepresentation of V isomorphic to k^n. Conclude by choosing a basis in $E_{11}V$.

Exercise 6. (PS 6). Suppose $H \subset G$ is a normal subgroup of a finite group, and $\rho : G/H \to \mathrm{Aut(V)}$ is a representation of G/H. Let $\phi : G \to G/H$ be the natural surjective homomorphism. Check that $\tilde{\rho} = \rho \circ \phi$ defines a representation of G in V. If ρ is irreducible, show that $\tilde{\rho}$ is irreducible as well. Show that inequivalent representations of G/H lift to inequivalent representations of G.

Exercise 7. (PS 8). Let V be an n-dimensional complex vector space. Then $GL(V)$ acts in the space $\wedge^m(V)$ by $g\cdot(v_1 \wedge v_2 \wedge \ldots \wedge v_m) = gv_1 \wedge gv_2 \wedge \ldots gv_m$, where $m \leq n$, and on the space $S^k(V)$ by $g\cdot(u_1 u_2 \ldots u_k) = (gu_1)(gu_2)...(gu_k)$.

(a) Show that $\wedge^m(V)$ is an irreducible representation of $GL(V)$, $m \leq n$.

(b) Show that $S^2(V)$ is an irreducible representation of $GL(V)$. In fact, a stronger statement holds: $S^m(V)$ is an irreducible representation of $GL(V)$ for any $m \geq 1$.

Exercise 8. (PS 11). Let $D_3 = \langle r, s \mid r^3 = 1, s^2 = 1, srs = r^{-1}\rangle$ be the dihedral group of order 6. Describe the irreducible complex representations of D_3 and compute its character table. Solution: the character table is given by

	(1)	(r, r^2)	(s, sr, sr^2)
V_0	1	1	1
V_s	1	1	-1
V_2	2	-1	0

Exercise 9. (PS 13). Let V_λ denote the Specht module for S_n, where λ is a partition of n.

(a) Show that

$$\operatorname{Res}^{S_n}_{S_{n-1}} V_\lambda \simeq \bigoplus_{\mu \in R(\lambda)} V_\mu,$$

where $R(\lambda)$ is the set of Young diagrams obtained by removing one square from Y_λ.

(b) Show that

$$\operatorname{Ind}^{S_n}_{S_{n-1}} V_\mu \simeq \bigoplus_{\lambda \in A(\mu)} V_\lambda,$$

where $A(\mu)$ is the set of Young diagrams obtained by adding one square from Y_μ.

Exercise 10. (PS 13). (Transitivity of the induction) Let $K \subset H \subset G$ be subgroups of a finite group G and V a complex representation of K. Show that

$$\operatorname{Ind}^G_H \operatorname{Ind}^H_K V \simeq \operatorname{Ind}^G_K V.$$