November 19, 2024

Problem Set 9 Solutions

Exercise 1. Let V be a complex vector space of dimension k and consider $V^{\otimes n}$. Then $V^{\otimes n}$ is a representation of the symmetric group defined as

$$\phi(\sigma)\left(u_1 \otimes u_2 \otimes \ldots \otimes u_n\right) = u_{\sigma(1)} \otimes u_{\sigma(2)} \ldots \otimes u_{\sigma(n)}.$$

By Maschke's theorem, $V^{\otimes n}$ decomposes as a direct sum of S_n- irreducible representations. The following exercise identifies the subspaces of $V^{\otimes n}$ that carry the trivial (resp. sign) isotypical component with respect to the S_n action.

(a) Define $P_+ : V^{\otimes n} \to V^{\otimes n}$ by

$$P_+(u_1 \otimes u_2 \otimes \ldots \otimes u_n) = \frac{1}{n!} \sum_{\sigma \in S_n} u_{\sigma(1)} \otimes u_{\sigma(2)} \ldots \otimes u_{\sigma(n)}.$$

Show that $P_+(V^{\otimes n}) \subset T_+$, where T_+ is the trivial isotypical component in $V^{\otimes n}$ with respect to the action of S_n (the largest submodule of $V^{\otimes n}$ isomorphic to a direct sum of trivial representations of S_n).

(b) Define $P_- : V^{\otimes n} \to V^{\otimes n}$ by

$$P_-(u_1 \otimes u_2 \otimes \ldots \otimes u_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) u_{\sigma(1)} \otimes u_{\sigma(2)} \ldots \otimes u_{\sigma(n)},$$

where $\varepsilon(\sigma)$ is the sign of the permutation σ. Show that $P_-(V^{\otimes n}) \subset T_-$, where T_- is the isotypical component of the sign representation in $V^{\otimes n}$ (the largest submodule of $V^{\otimes n}$ isomorphic to a direct sum of sign representations of S_n).

(c) Show that $P_+|_{T_+} = \mathrm{id}_{T_+}$, $P_-|_{T_-} = \mathrm{id}_{T_-}$ and deduce that P_+, P_- are projectors onto T_+, T_-.

(d) Show that $T_+ \cong S^n(V)$ and $T_- \cong \wedge^n(V)$.

Solution 1. We can view the tensor product $V^{\otimes n}$ as a $\mathbb{C}[S_n]$-representation, where $\tau \in S_n$ acts via $\tau \cdot u = \phi(\tau)(u)$ for $u \in V^{\otimes n}$.

(a) For $u \in V^{\otimes n}$ and $\tau \in S_n$, we have

$$\tau \cdot P_+(u) = \tau \cdot \left(\frac{1}{n!} \cdot \sum_{\sigma \in S_n} \sigma \cdot u\right) = \frac{1}{n!} \cdot \sum_{\sigma \in S_n} \tau\sigma \cdot u = \frac{1}{n!} \cdot \sum_{\sigma \in S_n} \sigma \cdot u = P_+(u)$$

because the map $\sigma \mapsto \tau\sigma$ is a bijection on S_n. It follows that $P_+(u)$ belongs to the isotypical component of the trivial $\mathbb{C}[S_n]$-representation of $V^{\otimes n}$, so $P_+(u) \subseteq \mathrm{span}_{\mathbb{C}}\{P_+(u)\} \subseteq T_+$, as required.

(b) As in part (a), we compute that

$$\tau \cdot P_-(u) = \frac{1}{n!} \cdot \sum_{\sigma \in S_n} \varepsilon(\sigma) \cdot \tau\sigma \cdot u = \varepsilon(\tau) \cdot \frac{1}{n!} \cdot \sum_{\sigma \in S_n} \varepsilon(\tau\sigma) \cdot \tau\sigma \cdot u = \varepsilon(\tau) \cdot \frac{1}{n!} \cdot \sum_{\sigma \in S_n} \varepsilon(\sigma) \cdot \sigma \cdot u = \varepsilon(\tau) \cdot P_-(u),$$

using in the second step that ε is a group homomorphism and in the third step that $\sigma \mapsto \tau\sigma$ is a bijection. Again, we conclude that $P_-(u)$ belongs to the isotypical component of the sign representation of $\mathbb{C}[S_n]$ in $V^{\otimes n}$, so $P_-(u) \in T_-$ as required.

(c) If $u \in T_+$ then $\sigma \cdot u = u$ for all $\sigma \in S_n$ and therefore

$$P_+(u) = \frac{1}{n!} \cdot \sum_{\sigma \in S_n} \sigma \cdot u = \frac{1}{n!} \cdot \sum_{\sigma \in S_n} u = \frac{1}{n!} \cdot n! \cdot u = u$$

as S_n has order $n!$. Analogously, we have $\sigma \cdot u = \varepsilon(\sigma) \cdot u$ for all $\sigma \in S_n$ and $u \in T_-$ and therefore

$$P_-(u) = \frac{1}{n!} \cdot \sum_{\sigma \in S_n} \varepsilon(\sigma) \cdot \sigma \cdot u = \frac{1}{n!} \cdot \sum_{\sigma \in S_n} \varepsilon(\sigma)^2 \cdot u = u$$

as S_n has order $n!$ and $\varepsilon(\sigma) \in \{\pm 1\}$ for all $\sigma \in S_n$. It follows that $P_+|_{T_+} = \mathrm{id}_{T_+}$ and $P_-|_{T_-} = \mathrm{id}_{T_-}$. As the image of P_+ is contained in T_+ and the image of P_- is contained in T_-, we further conclude that $P_+^2 = P_+$ and $P_-^2 = P_-$, so P_+ and P_- are projections onto T_+ and T_-, respectively.

(d) Recall that $S^n V$ is the quotient of $V^{\otimes n}$ by the subspace $U = \mathrm{span}_{\mathbb{C}}\{u - \sigma \cdot u \mid u \in V^{\otimes n}, \sigma \in S_n\}$ and write $\pi \colon V^{\otimes n} \to S^n V$ for the canonical quotient homomorphism. For $u \in V^{\otimes n}$ and $\sigma \in S_n$, we have

$$P_+(u - \sigma \cdot u) = P_+(u) - P_+(\sigma(u)) = P_+(u) - P_+(u) = 0$$

by a computation similar to part (a). It follows that $U \subseteq \ker(P_+)$, so P_+ induces a homomorphism $\bar{P}_+ \colon S^n V \to T_+$ with $P_+ = \bar{P}_+ \circ \pi$. Then

$$\bar{P}_+ \circ (\pi|_{T_+}) = \left(\bar{P}_+ \circ \pi\right)\big|_{T_+} = P_+|_{T_+} = \mathrm{id}_{T_+}$$

by part (c). Furthermore, we have $\pi \circ P_+ = \pi$ because $\pi(\sigma \cdot u) = \pi(u)$ for all $\sigma \in S_n$ and $u \in V^{\otimes n}$ and it follows that

$$(\pi|_{T_+}) \circ \bar{P}_+ \circ \pi = \pi \circ P_+ = \pi.$$

As π is surjective, this implies that $(\pi|_{T_+}) \circ \bar{P}_+ = \mathrm{id}_{S^n V}$, so $\bar{P}_+$ is an isomorphism with inverse $\pi|_{T_+}$.

Analogously, $\bigwedge^n V$ is the quotient of $V^{\otimes n}$ by the subspace

$$U' = \mathrm{span}_{\mathbb{C}}\{u \mid u \in V^{\otimes n} \text{ such that } u = s \cdot u \text{ for some transposition } s \in S_n\}$$

and we write $\pi' \colon V^{\otimes n} \to \bigwedge^n V$ for the canonical quotient homomorphism. For $u \in V^{\otimes n}$ such that $u = s \cdot u$ for some transposition $s \in S_n$ then

$$P_-(u) = P_-(s \cdot u) = \varepsilon(s) \cdot P_-(u) = -P_-(u)$$

and it follows that $P_-(u) = 0$, so $U' \subseteq \ker(P_-)$. As before, we obtain a homomorphism $\bar{P}_- \colon \bigwedge^n V \to T_-$ with $P_- = \bar{P}_- \circ \pi'$ and using part (c), we see that

$$\bar{P}_- \circ (\pi'|_{T_-}) = \left(\bar{P}_- \circ \pi\right)\big|_{T_-} = P_-|_{T_-} = \mathrm{id}_{T_-}.$$

Now we have $\pi'(\sigma \cdot u) = \varepsilon(\sigma) \cdot \pi'(v)$ for all $\sigma \in S_n$ and $u \in V^{\otimes n}$ and therefore $\pi' \circ P_- = \pi'$. As before, it follows that

$$(\pi'|_{T_-}) \circ \bar{P}_- \circ \pi' = \pi' \circ P_- = \pi'$$

and surjectivity of π' implies that $(\pi'|_{T_-}) \circ \bar{P}_- = \mathrm{id}_{\bigwedge^n V}$. Hence $\bar{P}_-$ is an isomorphism with inverse $\pi'|_{T_-}$.

Exercise 2. (Hilbert's third problem)
This exercise shows how tensor products can be used to solve a problem in 3-dimensional geometry.

(a) Define the Dehn invariant $D(A) \in \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q}$ of a polyhedron A by

$$D(A) = \sum_a l(a) \otimes \frac{\beta(a)}{\pi},$$

where the sum is taken over the edges of A, $l(a)$ is the length of a, and $\beta(a)$ is the angle at the edge a. Show that if you cut A into B and C by a straight cut, then $D(A) = D(B) + D(C)$.

(b) Find the angle α at the edge of a regular tetrahedron and prove that it is not a rational multiple of π.
Hint: Assume that $\alpha = \frac{m}{n}\pi$ and deduce that $(x + x^{-1}) = \frac{2}{3}$, where x is a root of unity of degree n. Show by induction that $x^k + x^{-k}$ has denominator 3^k and deduce a contradiction.

(c) Compute the Dehn invariants of a cube and of a regular tetrahedron and conclude that a cube cannot be cut with straight cuts and rebuilt in the shape of a regular tetrahedron of the same volume.

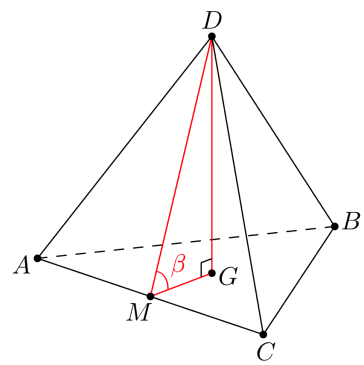
Solution 2. (a) We first have to see in how many different ways a straight cut can change the Dehn invariant. For an edge a of A, the pair $(l(a), \beta(a))$ changes after the cut either if $l(a)$ changed (the cut cuts an edge transversely, dividing the edge in two) or if $\beta(a)$ changed (the cut cuts along the edge, dividing the angle in two). On the other hand, new edges $(l(a), \beta(a))$ are created every time the cut divides a face in two, so there are in total three cases to consider.

(i) If an edge (l, β) is cut transversely, it is divided in two pairs (l_1, β) and (l_2, β) with $l_1 + l_2 = l$. Then $l \otimes \beta/\pi = l_1 \otimes \beta/\pi + l_2 \otimes \beta/\pi$, leaving the invariant unchanged.

(ii) If an edge (l, β) is cut longitudinally, it is divided in two pairs (l, β_1) and (l, β_2) with $\beta_1 + \beta_2 = \beta$. Then $l \otimes \beta/\pi = l \otimes \beta_1/\pi + l \otimes \beta_2/\pi$, leaving the invariant unchanged.

(iii) If the cut divides a face in two, it will create two pairs with supplementary angles (l, β) and $(l, \pi - \beta)$ but together they will not change the invariant: $l \otimes \beta/\pi + l \otimes (\pi - \beta)/\pi = l \otimes 1 = 0$, as $1 \in \mathbb{Q}$.

(b) We first claim that the angle is $\beta = \arccos(1/3)$. Indeed, let $ABCD$ be a regular tetrahedron, G the center of ABC, and M the midpoint of AC.



As G is the barycenter of ABC, we have $MG = MB/3$, so that $\cos(\beta) = MG/MD = 1/3 \cdot MB/MD = 1/3$.

Now assume that $\arccos(1/3)/\pi = \frac{m}{n}$ is a rational number. Then,

$$\frac{1}{3} = \cos\left(\frac{m\pi}{n}\right) = \frac{1}{2}\left(e^{i\frac{m\pi}{n}} + e^{-i\frac{m\pi}{n}}\right) = \frac{1}{2}(x + x^{-1}),$$

where $x = e^{i\frac{m\pi}{n}}$ is a roots of unity of order $2n$. We now claim that for $k \geq 0$, $x^k + x^{-k}$ is a rational number with denominator 3^k. We prove it by induction on k, as the case $k = 0$ is clear and the case $k = 1$ is already covered above. Assume that $x^i + x^{-i}$ has denominator 3^i for $1 \leq i < k$. Then,

$$x^{k+1} + x^{-(k+1)} = (x + x^{-1})(x^k + x^{-k}) - (x^{k-1} + x^{-k+1}) = \frac{2}{3}(x^k + x^{-k}) - (x^{k-1} + x^{-(k-1)}),$$

proves the induction step, inspecting the denominator of the right hand side. On the other hand, since $x^{2n} = 1$, we have $x^{2n} + x^{-2n} = 2$, a contradiction. Thus, $\beta = \arccos(1/3)$ is not a rational multiple of π.

(c) Since the dihedral angle at an edge of a cube is $\pi/2$, we have for a cube $\beta(a)/\pi = \frac{1}{2} \equiv 0 \mod \mathbb{Q}$, and therefore $D(C) = 0$. On the other hand, for a tetrahedron we have $D(T) = 6l \otimes \arccos(1/3)/\pi \neq 0$. Since $D(C) \neq D(T)$, one of them cannot be cut and reconstructed to obtain the other.